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The True Story Behind Frattini's Argument *

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In his memoir [7] *Intorno alla generazione dei gruppi di operazioni* (1885), **Giovanni Frattini** (1852–1925) introduced the idea of what is now

known as a *non-generating element* of a (finite) group. He proved that the set of all non-generating elements of a group G constitutes a normal subgroup of G, which he named Φ. This subgroup is today called the *Frattini subgroup* of G. The earliest occurrence of this denomination in literature traces back to a paper of Reinhold Baer [1] submitted on September 5th, 1952. Frattini went on proving that Φ is nilpotent by making use of an insightful, renowned argument, the intellectual ownership of which is the subject of this note. We are talking about what is nowadays known as *Frattini's Argument*.

Giovanni Frattini was born in Rome on January 8th, 1852 to Gabriele and Maddalena Cenciarelli. In 1869, he enrolled for Mathematics at the University of Rome, where he graduated in 1875 under the

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supervision of Giuseppe Battaglini (1826–1894), together with other up-and-coming mathematicians such as Alfredo Capelli. Since October 1876, he began his teaching career in secondary school. His excellent teaching qualities made him one of the most esteemed mathematics teachers of his times, and his textbooks, among which we recall "Aritmetica Pratica" [8] for primary schools, were widely used. Frattini retired from teaching in 1921 and died on July 21st, 1925. A more exhaustive account of the life of Giovanni Frattini can be found in [5],[6],[10].

We present here a passage from the quoted paper where Frattini used the Argument for the first time.(1)

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5. «The group Φ is a Capelli's Ω_0-group (1). [...]

«There is the following theorem: A subgroup Γ, exceptional in [a group] G, can always efficaciously contribute to the generation of G,(2) when there are in Γ at least two different groups among those having order the biggest power of some of the prime factors of the order of Γ.(3)

«Before proving this theorem, I notice that it is essentially due to Capelli, who proves ([2]) that under the aforesaid assumption there are subgroups of G contributing with their substitutions to all the periods(4) of Γ.(5) In order to prove the equivalence of the two propositions, we will indeed point out that, if Γ is exceptional in G, every subgroup L of G permutes with Γ ([3]), so that putting together the periods of Γ having substitutions in common with a

(1) For an explanation on the terminology of the excerpts, see the dedicated section. Notice also that the black numbered footnotes are due to the authors themselves, while the red numbered ones contain our comments.

(2) Γ *efficaciously contributes* to the generation of G when there exists a proper subgroup K of G such that $\langle K, \Gamma\rangle = G$.

(3) A more contemporary reading of this theorem is the following. Let G be a finite group and N a normal subgroup of G. If N contains two distinct Sylow p-subgroups for a given prime p, then N cannot be contained in $\Phi(G)$ (see note(12)).

(4) The sentence *contributing to the periods of a given subgroup* Γ *of a group* G stands for *having non-trivial intersection with all the cosets of* Γ *in* G.

(5) The proof of this result (pp. 263–264 of the quoted paper of Capelli) uses Frattini's Argument. Moreover, the proof shows that the subgroups contributing to all periods of Γ are the normalizers of the distinct Sylow p-subgroups of Γ.

period of L, a new distribution in periods of the substitutions of G will take place and, precisely, the distribution pertaining the group generated by Γ and L[(6)] as I have proved in my Memoir: *Intorno ad alcune proposizioni della teoria delle sostituzioni* ([4]). This evidently reveals that: a necessary and sufficient condition so that there are subgroups L of G contributing with their substitutions to all the periods of Γ is that Γ generates G together with some subgroup of G and smaller than G,[(7)] namely that Γ efficaciously contributes to the generation of G with some system of substitutions extraneous to Γ.

«That being said, let us come to the proof of the stated theorem. Let P be one of the subgroups of order p^α (the biggest α) contained in Γ, and S a substitution of G. Let us say P' the group of order p^α (contained in Γ) which P is transformed into by S.

«We know there are substitutions in Γ which transform P into P'. Let γ be one of these. The substitution $S\gamma^{-1} = \sigma$ will belong to the group of substitutions of G which transform P into itself, and we will have: $S = \sigma\gamma$.

«Being S any substitution of G, we conclude that the group Γ and that of the substitutions transforming P into itself generate G. Now the group Γ efficaciously contributes to this generation, provided that the substitutions of G transforming P into itself do not form the whole G. But in

(6) In other words, let $g_1\Gamma, \ldots, g_n\Gamma$ and $h_1L, \ldots, h_mL$ be the cosets in G of Γ and L, respectively. Moreover, let

$$M'_{h_i} = \{g_j\Gamma | g_j\Gamma \cap h_iL \neq \emptyset\}$$

and put

$$M_{h_i} = \bigcup_{H \in M'_{h_i}} H.$$

It is stated here that $M_{h_i} = h_iL\Gamma$.

(7) Let us state this in a more modern fashion. *Let* G *be a group and* Γ *be a proper subgroup of* G *containing two distinct Sylow* p*-subgroups for a given prime* p*. There exists a proper subgroup* L *of* G *having non-empty intersection with each coset of* Γ *in* G *if and only if there exists a proper subgroup* K *such that* $\Gamma K = G$. The necessary condition is trivial, since $\Gamma L = G$. As for the sufficient condition, since $G = \Gamma K = M_1$ (see note (6)), it follows that K must have non-trivial intersection with all cosets of Γ in G.

this case P' would coincide with P and there would not be in Γ two distinct subgroups of order p^α.[(8)]

«And now, since the group Φ, which is exceptional in G, cannot, owing to its definition, efficaciously contribute to the generation of G, there will not be in Φ two distinct groups of orders p^α, $q^\beta, \ldots$, respectively. The group Φ will hence be an Ω_0-group.[(9)]

([1]) I name Capelli's Ω_0-groups those groups which, being of order $p^\alpha.q^\beta.r^\gamma\ldots$, do contain nothing but one group of order: $p^\alpha, q^\beta, r^\gamma, \ldots$, respectively,[(10)] since Capelli, in his Memoir: *Sopra la composizione di gruppi di sostituzioni* (R. Accademia dei Lincei, v. XIX),[(11)] has proved many properties concerning these groups, and among the others the following two properties: The composition factors of Ω_0-groups are prime numbers: Every subgroup of an Ω_0-group is itself an Ω_0-group. Combining this second property with our fourth proposition,[(12)] it will easily follow that: If it is not possible for the fundamental group to be generated by a given subgroup combined with any of the others, the first subgroup will belong to the species of Capelli's Ω_0-groups.

([2]) loc. cit.

([3]) For all α and β, it takes place a relation of the form: $l_\alpha.\gamma_\beta = \gamma'_\beta.l'_\alpha$.

([4]) Mem. of the R. Accademia dei Lincei, v. XVIII, 1883–84.

This passage is often quoted by several authors to attribute the Argument to Frattini: the earliest occurrence of this attribution can be found in Wolfgang Gaschütz [9] in 1953 (*mit einer eleganten Schlußweise*), later in Baer [2] in 1956 (*the arguments used in the proofs of results of this section are due to Frattini*), in Bertram Huppert [12] in 1967

(8) This concludes the proof of the stated theorem.

(9) Proposition 5 is hence proved.

(10) Ω_0-groups are nothing but finite nilpotent groups.

(11) Communicated on March 2nd, 1884.

(12) *In order that the fundamental group can be generated by a given subgroup combined with some of the others, it is a necessary and sufficient condition that the first is not totally formed with substitutions of* Φ. In other words, a subgroup K of a group G is contained in $\Phi(G)$ if and only if, for each $K' \leqslant G$ such that $\langle K, K'\rangle = G$, it follows that $K' = G$.

(*sog.* Frattini-*Argument*), in Daniel Gorenstein [11] in 1968 (*sometimes called Frattini-Argument*), and so on.

However, one should have noticed that the Argument is actually used to prove a result essentially due to Capelli, as Frattini himself pointed out, and if one checks the proof of that theorem, it turns out, as we will see in a moment, that the real father of the so-called Frattini's Argument is Alfredo Capelli, indeed. This fact is not unknown, but it passed unnoticed, since some papers on the subject were published mostly for an Italian audience (see [4],[10]) and some others aimed mostly at historians of mathematics or were not focused on this topic (see, for instance, [13]).

It is hence probably due to the authority and prestige of such great mathematicians that the erroneous expression *Frattini's Argument* entered the common lexicon of modern group theory. Therefore, our aim here is to shed new light on the subject. In order to do this, we have presented a faithful translation of Frattini's misconceived paper and now we want to rediscover and revalue the very first uses of what should be called *Capelli's Argument*, according to the suggestion made in [4].

Alfredo Capelli was born in Milan on August 5th, 1855 to Arminio and Gioconda Manfardi. He studied Mathematics at the University of Rome. Here, among others, his teachers were Luigi Cremona (1830–1903), Eugenio Beltrami (1835–1900) and G. Battaglini. He graduated in 1877 under the supervision of Battaglini. Then, he moved to Pavia, Berlin, Palermo and finally Naples where he stayed until his death by heart attack on January 28th, 1910. During his mathematical career, he published over 80 papers giving substantial results on the theory of algebraic forms, on group theory and on the theory of algebraic equations. A more exhaustive account of the life of Alfredo Capelli can be found in [4], [13],[16],[17],[18].

The following piece is taken from the paper [3] *Sopra la composizione dei gruppi di sostituzioni* (1884) by Capelli and it constitutes the earliest evidence of the so-called *Frattini's Argument*. The first part of the paper revolves around the following problem:

> *Let* G *be a finite group and* H *a normal subgroup of* G. *How can we choose a system of representatives of* H *in* G *such that the subgroup they generate has smallest order?*

Capelli first proved that for such a Γ, the intersection $\Gamma_0 = \Gamma \cap H$ has to contain only one Sylow p-subgroup for each prime p, namely that Γ_0 is nilpotent. This is proved by means of the following theorem:

> *Let* G *be a finite group and* H *a normal subgroup of* G *containing two distinct Sylow* p*-subgroups for a given prime* p*. Then there exists a proper subgroup* Γ *of* G *having non-trivial intersection with each coset of* H *in* G*, i.e.* $G = H\Gamma$.

The first evidence of Frattini's Argument is actually shown in the proof of this result. What follows is a translation of the subsection containing it.

> 3. If p is one of the prime factors of the order ν of the group H, and p^α is the largest power of p dividing ν, it is known that in H there are partial groups $P_0, P_1, P_2, \ldots$ of order p^α (1).[13] They are all obtained from one of them, for the sake of simplicity from P_0, transforming P_0 by substitutions of H, so that one can always put:
>
> $$P_1 = h'^{-1} P_0 h', \quad P_2 = h''^{-1} P_0 h'', \ldots. \tag{2}$$
>
> where $h', h'', \ldots$ are substitutions of H. Let Π be the group made by all those substitutions of G transforming the group P_0 into itself. I say that every period[14] of G has some substitution in common with Π. Let us consider, in fact, any period:
>
> $$G_i,\ G_i h_2,\ \ldots\ldots,\ G_i h_\nu \text{[15]} \tag{3}$$
>
> and let us transform P_0 by the substitution G_i. Since G_i transforms the group H, which by assumption is permutable with all substitutions of G, into itself, it will transform the group P_0, which belongs to H, into a group $G_i^{-1} P_0 G_i$ equally contained in H. And since this new group is still

[13] The subgroups $P_0, P_1, P_2, \ldots$ are of course the Sylow p-subgroups of H.
[14] He is meaning the cosets of H in G.
[15] $\{1, h_2, \ldots, h_\nu\} = H$ and G_i is a given element of G.

of order p^α, it will belong to the series $P_0, P_1, \ldots$; put, for the sake of simplicity:

$$G_i^{-1} P_0 G_i = P_\varepsilon \ .$$

By (2) one will then have:

$$G_i^{-1} P_0 G_i = h^{(\varepsilon)-1} P_0 h^{(\varepsilon)}$$

thence also:

$$h^{(\varepsilon)} G_i^{-1} P_0 G_i h^{(\varepsilon)-1} = P_0$$

which can also be written:

$$(G_i h^{(\varepsilon)-1})^{-1} P_0 (G_i h^{(\varepsilon)-1}) = P_0 \, .$$

The group P_0 is so transformed into itself by the substitution $G_i h^{(\varepsilon)-1}$, which clearly belongs to the substitutions (3), which compose the i^{th} period. It is hence proved what has been stated, and, if ν' is the order of the group Π_0 formed by all substitutions of H transforming the group P_0 into itself, one concludes that every period of G contains ν' substitutions equally transforming P_0 into itself. It is therefore established that generally there exist partial groups contained in G and containing substitutions of each period of G, since the group Π could not coincide with the whole group G unless the group Π_0 coincides with the whole H, namely in the particular case in which P_0 is transformed into itself by all substitutions of H,[(16)] or, which is the same, in the case in which H contains only one group of order p^α.

If this happens to be the case, one shall start instead from a group Q_0[(17)] of order q^β, and if H contains other groups

(16) Capelli is proving here that $N_H(P_0) = H$ if and only if $N_G(P_0) = G$. The sufficiency is clear. In order to prove the necessity of the statement, the author shows that each coset of H in G contains $\nu = \nu'$ elements normalizing P_0. Since the order of H is ν, it follows that each coset of H in G normalizes P_0, which is hence normal in G.

(17) Q_0 is Sylow q-subgroup of H for a prime $q \neq p$.

of order q^β, one will conclude as above that the group made by the substitutions of G transforming Q_0 into itself has order smaller than that of G and contains substitutions of each period of G. Proceeding in this way, the following theorem is established: There always exists a partial group of G containing substitutions of all periods of G, every time that the first period H[18] contains at least two different groups having order the biggest power of a certain prime number.

From this it follows that, if Γ is one of the groups of smallest order satisfying the problem,[19] the group Γ_0 will contain only one group having as order the largest possible power of each prime number, since if this were not the case it would contain a partial group of smaller order equally satisfying the problem.

([1]) In another paper (*Sopra l'isomorfismo dei gruppi di sostituzioni.*[20] Giornale di Mat. Tom. XVI) we have established this theorem in a direct way by means of isomorphism observations ignoring that Mr. Sylow had come to it before us (Math. Annalen. Tomo V),[21] using the theorem of Cauchy saying that a group of substitutions whose order is divisible for the prime number p contains at least one substitution of order p. We hence use the current occasion to remedy the missed quote of this author's name. Concerning the proofs, we also refer to the work of Mr. Netto[22] *Substitutionentheorie und ihre Anwendung auf die Algebra*. Leipzig, 1882.

At the end of the paper, during the third part, a second and last application of Frattini's Argument makes its appearance. This part of the paper is devoted to prove that finite nilpotent groups are the only finite groups having the *normalizer property*, i.e. the requirement that each proper subgroup is properly contained in its normalizer.

That a finite nilpotent group must satisfy the normalizer property is proved in subsection 11 of the quoted article. Frattini's Argument

(18) Interestingly enough, Capelli chose this roundabout expression to name the subgroup H.

(19) See the introduction to this excerpt.

(20) Published in 1878.

(21) *Théorèmes sur les groupes de substitutions*, 1872.

(22) Eugen Netto (1848–1919).

is actually used in proving the converse implication. A translation of this fragment is given below.

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12. [...] [The] group U will not be of type Ω_0,[(23)] therefore at least one of the groups P, Q, ... contained in U, of orders p^α, q^β, ...[(24)] will not be the unique of its order. For the sake of simplicity, let P be one of these, and take V to be the group made by all substitutions of U transforming P into itself. It is necessarily a partial group, since, if it coincided with U, this would mean that P is transformed into itself by each substitution of U, and so, as it is known, it will be the only group of its order contained in U. [...] [Suppose by a contradiction that two of the subgroups:]

$$V,\ \sigma_1^{-1}V\sigma_1,\ \sigma_2^{-1}V\sigma_2,\ \ldots,\ \sigma_{\lambda-1}^{-1}V\sigma_{\lambda-1}\text{[(25)]}$$

coincide,[(26)] for the sake of simplicity, let:

$$\sigma_i^{-1}V\sigma_i = \sigma_j^{-1}V\sigma_j.$$

Hence, one would deduce that:

$$\sigma_j\sigma_i^{-1}V\sigma_i\sigma_j^{-1} = V$$

or, which is the same:

$$\tau^{-1}V\tau = V$$

setting for brevity:

$$\sigma_i\sigma_j^{-1} = \tau.$$

(23) Ω_0-groups are defined in the last part of subsection 5 as those finite groups having only one Sylow p-subgroup for each prime p. In other words, they are precisely the finite nilpotent groups. See note ([1]) of Frattini's passage, where the author refers to Capelli's paper.

(24) Here, $P, Q, \ldots$ are Sylow subgroups of order p^α, q^β, ..., respectively.

(25) $1, \sigma_1, \sigma_2, \ldots, \sigma_{\lambda-1}$ are defined in subsection 10, scheme (10), as the elements of a complete system of representatives for the right cosets of V in U.

(26) Notice that if they do not coincide, then V coincides with its normalizer in U, which only occurs when $V = U$, namely when P is the unique Sylow p-subgroup of U.

It can be easily recognized that the substitution τ cannot belong to V, since if this were the case:

$$\sigma_i \sigma_j^{-1} = \nu$$

where ν is a substitution of V, one would deduce from it:

$$\sigma_i = \nu \sigma_j$$

and:

$$V\sigma_i = V\nu\sigma_j = V\sigma_j$$

which is contrary to the construction of scheme (10).[27]

Since now τ transforms the group V into itself, the partial group P will be transformed into a group $\tau^{-1}P\tau$ as well contained in U.[28] However, all groups of order p^α contained in V can be obtained transforming P by substitutions of V; we will hence have:

$$\tau^{-1}P\tau = \nu_i^{-1}P\nu_i$$

where ν_i is a certain substitution of V. From here we learn:

$$(\tau\nu_i^{-1})^{-1}P(\tau\nu_i^{-1}) = P$$

thence, since the substitutions of V are precisely the ones transforming P into itself, we will necessarily get:

$$\tau\nu_i^{-1} = \nu_j \text{[29]}$$

and:

$$\tau = \nu_j\nu_i.$$

The substitution τ hence belongs to the group V, which is in contradiction[30] with what we have seen a short while ago.

[27] See note (25).
[28] This is an obvious misprint: U should be V.
[29] Here ν_j denotes an element of V.
[30] As pointed out in [4], this contradiction can be easily reached noticing that P is the unique Sylow p-subgroup of V.

> [...] We conclude that: the groups of type Ω_0 are just those satisfying the property of having each partial group being transformed into itself by some substitution of the group not contained in the partial group itself.

Finally, in these last few lines, we would like to draw reader's attention to one of the possibly main reasons for the paper of Capelli having passed unnoticed. Following Georg Abram Miller [14],(31) it seems that the review of Capelli's note by Eugen Netto published in *Jahrbuch über die Fortschritte der Mathematik* (v. XVI, p. 116) attributed to Capelli a theorem which is evidently incorrect. Because of the authority of Netto and since the reviews in this journal were so widely read and so frequently referred to, this could have led to the apparent neglect of this paper. Netto was not new to this kind of unfair attitude: wrong criticisms or inadmissible omissions can for instance be found in his reviews of the works of Camille Jordan and Ludwig Sylow (see also [15]).

On the account of the produced proofs, we highly believe and wish that the contributions of Capelli to group theory should be at least revalued, giving him the credit he deserves.

A Note on Terminology

Being the excerpts translated from a language which is now outdated, we are now presenting, for reader's convenience, a brief explanatory note on the used terminology.

All groups considered are *groups of substitutions* or *groups of operations*; however this is only due to historical reasons. One can just read *groups* in place of these formulations, viewing the substitutions (operations) just as ordinary elements of a group.

When dealing with a group G and a (proper) subgroup H, the former is usually called the *fundamental group* and the latter a (*partial) group*. It is also usual to refer to a proper subgroup just as a subgroup.

For a subgroup, the terms *exceptional* and *period* are used instead of *normal* and *coset*, respectively.

The action of conjugating by an element of G is referred to as *transforming*.

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(31) Notice that Miller was among the first ones to recognize the worth of Capelli's work in group theory.

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