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Complete Groups of Order $3p^6$

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Abstract

For each prime number p with $3 \mid p-1$, we construct a group of order $3p^5$, whose automorphism group is a complete group of order $3p^6$.

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1 Introduction

Several papers in the literature deal with the construction of complete groups of odd order ([2],[12],[13]). There are infinitely many such groups ([4],[8]), but the smallest hitherto published example seems to be one of order $5 \cdot 3^{12}$ constructed by Heineken [6]. We shall prove the following result.

Theorem *Let p be a prime number with $3 \mid p-1$. Then there is a group H with trivial centre $\mathbb{Z}(H) = 1$, and automorphism group $G = \operatorname{Aut} H$, such that H has order $|H| = 3p^5$, G has order $|G| = 3p^6$, and G is complete.*

Remark (a) The group G is supersoluble, so this demonstrates the pertinence of Heineken's remark [6] that: "It remains to review supersoluble groups".

(b) In [5] Hegarty and MacHale pose some questions:

(i) What is the smallest order of a non-nilpotent group with an automorphism group of odd order?

(ii) What is the smallest order of a non-nilpotent automorphism group of odd order?

(iii) What is the smallest possible order of a complete group of odd order?

We conjecture that when $p = 7$, the group H constructed here of order $3 \cdot 7^5$ provides an answer to (i), and we also conjecture that the group G of order $3 \cdot 7^6$ answers (ii). Moreover the second author has proved [3] that the answer to (iii) is indeed $3 \cdot 7^6$.

(c) Clearly the key to the Theorem is the Sylow p-subgroup of H, so its properties are of interest. Let p be an odd prime number, and suppose Q is a nontrivial p-group, with order $|Q| = p^n \neq 1$ and nilpotency class c. Define $\Phi = Q^p Q'$ to be the Frattini subgroup of Q.

The arguments of Heineken and Liebeck [7, Sections 4 and 5] show that if either $n \in \{1,2,3,4\}$, or $n = 5$ and $c \in \{1,2\}$, then there exist an automorphism $\theta \in \operatorname{Aut} Q$, and a nontrivial element $g \in Q - \Phi$, such that $g^\theta = g^{-1}$. In this case θ has even order, so $2 \,\Big|\, |\operatorname{Aut} Q|$, and Q is an unlikely candidate for involvement in a complete group of odd order.

Next write $\mathbb{F}_p$ for the field of order p, and let k be an element in the multiplicative group $\mathbb{F}_p^\times = \mathbb{F}_p - 0$. Define $P = \langle x, x_1 \rangle$ to be the group with defining relations (1) below. The first author has proved that if $n = 5$ and $c \in \{3,4\}$, then again Q has an automorphism which inverts an element of $Q - \Phi$, except when $p \geqslant 5$ and $Q = P$ for some value of k [1, Theorems 2 and 3]. On the other hand [1, Theorem 4]

$$|\operatorname{Aut} P| = \begin{cases} p^6 & \text{when } p \geqslant 5 \text{ and } 3 \nmid p-1 \\ 3p^6 & \text{when } 3 \mid p-1 \end{cases}$$

Taking $p = 7$, we get $|\operatorname{Aut} P| = 3 \cdot 7^6$, so $\operatorname{Aut} P$ is an alternative conjectural answer to the question (ii) above (but is not complete).

Finally let k_0 be a primitive $(p-1)$-th root of 1 modulo p^2, and put $y = x^{k_0^i}$ and $y_1 = x_1^{k_0^{2i}}$. Then $x = y^{k_0^{-i}}$ and $x_1 = y_1^{k_0^{-2i}}$, so $P = \langle y, y_1 \rangle$, and using (1), together with Lemma (f) below, we get the relations

$$y^p = y_1^{p^2} = [y_1^p, y] = 1$$
$$y_1^{k_0^{3i}kp} = [y_1, y, y_1] = [y_1, y, y, y].$$

If $p \geqslant 5$ and $3 \nmid p-1$, then $\mathbb{F}_p^\times = \langle k_0^3 \rangle$, and hence the groups P are isomorphic to each other for all values of k. Similarly if $3 \mid p-1$, then

$$\mathbb{F}_p^\times = \langle k_0^3 \rangle \cup \langle k_0^3 \rangle k_0 \cup \langle k_0^3 \rangle k_0^2,$$

and hence each group P is isomorphic to one of the 3 groups got by taking $k = 1$, k_0 and k_0^2. Moreover, the list compiled by James [10] indicates that no two of these 3 groups are isomorphic to each other.

Construction — As above, let p be a prime number with $p \geqslant 5$, and choose an element $k \in \mathbb{F}_p^\times = \mathbb{F}_p - 0$. Take $P = \langle x, x_1 \rangle$ with defining relations

$$\left.\begin{array}{rcl} x^p &=& x_1^{p^2} = [x_1^p, x] = 1, \\ x_1^{kp} &=& [x_1, x, x_1] = [x_1, x, x, x] \end{array}\right\} \qquad (1)$$

Write P_i for the i-th term of the lower central series of P, and make the inductive definition

$$x_i = [x_{i-1}, x] \quad (i \geqslant 2).$$

Applying Jacobi's identity in the Lie ring $\bigoplus_{i \geqslant 1} P_i/P_{i+1}$ [9, III Aufgabe 8 (3), page 268], and working modulo P_5, we get

$$\begin{array}{rcl} 1 &\equiv& [x_1, x, [x_1, x]] \cdot [x, [x_1, x], x_1] \cdot [x_1, x, x_1, x] \\ &\equiv& 1 \cdot [x_1, x, x, x_1]^{-1} \cdot [x_1, x, x_1, x] \quad (\text{mod } P_5). \end{array}$$

Using the relations (1), together with this congruence, we deduce that

$$\begin{array}{ll} P &= \langle x, x_1 \rangle P_2, \\ P_2 &= \langle [x_1, x] \rangle P_3 = \langle x_2 \rangle P_3, \\ P_3 &= \langle [x_1, x, x],\ [x_1, x, x_1] \rangle P_4 = \langle [x_1, x, x] \rangle P_4 = \langle x_3 \rangle P_4, \end{array}$$

$$\begin{aligned} P_4 &= \langle [x_1, x, x, x],\ [x_1, x, x, x_1] \rangle P_5 = \langle [x_1, x, x, x],\ [x_1, x, x_1, x] \rangle P_5 \\ &= \langle [x_1, x, x, x] \rangle P_5 = \langle x_4 \rangle P_5, \\ P_5 &= \langle [x_1, x, x, x, x],\ [x_1, x, x, x, x_1] \rangle P_6 = P_6. \end{aligned}$$

Since P'/P_3 is cyclic, this implies that $P'' \leqslant [P', P_3] \leqslant P_5 = 1$. Moreover $P_i^p \leqslant P_{i+1}$ $(i \geqslant 1)$, so P is a group of maximal class [9, III (14.1)]. We shall use the following well known results.

Proposition (Rose [11, Corollary 1.2(iii)]) *Let* H *be a finite group with* $Z(H) = 1$, *and suppose* $H \leqslant G \leqslant \operatorname{Aut} H$. *Let* π *be a set of prime numbers, and suppose* $H = O^{\pi}(G)$ *is the smallest normal subgroup of* G *such that* G/H *is a* π*-group. Then* $\operatorname{Aut} G = N_{\operatorname{Aut} H}(G)$.

Lemma *Suppose* P *is a group with* $P'' = 1$, *and put* $P^* = P/P'$. *Let* $* : P \to P^*$ *be the natural homomorphism, and form the group ring* $\mathbb{Z}P^*$. *Suppose that* $r_1, r_2, \ldots, r_n$ *are natural numbers and* $y_1, y_2, \ldots, y_n$ *are elements of* P, *and take* $v_i = y_i^* - 1 \in \mathbb{Z}P^*$ $(1 \leqslant i \leqslant n)$.

(a) *Then* P' *can be regarded as a* $\mathbb{Z}P^*$*-module.*

(b) *If* $P = \langle x, x_1 \rangle$, *then* $P' = \{[x_1, x]^S : S \in \mathbb{Z}P^*\}$.

(c) *If* $z \in P'$ *and* $S \in \mathbb{Z}P^*$, *then*

$$[z^S, y_1, y_2, \ldots, y_n] = [z, y_1, y_2, \ldots, y_n]^S.$$

(d) *If* r *is a natural number, it follows that* $[y_1^r, y_2] = [y_1, y_2]^{S_1}$ *and* $[y_1, y_2^r] = [y_1, y_2]^{S_2}$, *with*

$$S_i = 1 + y_i + y_i^2 + \ldots + y_i^{r-1} \quad (i{=}1,2).$$

(e) *If* $n \geqslant 2$, *then* $[y_1^{r_1}, y_2^{r_2}, \ldots, y_n^{r_n}] = [y_1, y_2, \ldots, y_n]^{S_1 S_2 \ldots S_n}$, *with*

$$S_i = \sum_{s=1}^{r_i} \binom{r_i}{s} v_i^{s-1} \quad (1 \leqslant i \leqslant n).$$

(f) *If* $n \geqslant 2$, *and* $[y_1, y_2, \ldots, y_n]$ *is in the centre* $Z(P)$, *then*

$$[y_1^{r_1}, y_2^{r_2}, \ldots, y_n^{r_n}] = [y_1, y_2, \ldots, y_n]^{r_1 r_2 \ldots r_n}.$$

Proof — (a) This holds because $P' \leqslant C_P(P')$.

(b) This follows from (a), since P' is the normal subgroup of P generated by $[x_1, x]$.

(c) Using (a) we get

$$\begin{aligned}[z^S, y_1, y_2, \ldots, y_n] &= z^{Sv_1v_2\ldots v_n} = z^{v_1v_2\ldots v_n S} \\ &= [z, y_1, y_2, \ldots, y_n]^S.\end{aligned}$$

(d) This can be proved inductively, using the fact that if $r \geqslant 2$, then

$$\begin{aligned}[y_1^r, y_2] &= [y_1, y_2]^{y_1^{r-1}}[y_1^{r-1}, y_2], \\ [y_1, y_2^r] &= [y_1, y_2^{r-1}][y_1, y_2]^{y_2^{r-1}}.\end{aligned}$$

(e) Using (c) and (d) we get the required equation, where

$$\begin{aligned}S_i &= 1 + y_i + y_i^2 + \ldots + y_i^{r_i - 1} = \frac{y_i^{r_i} - 1}{y_i - 1} \\ &= \frac{(1+v_i)^{r_i} - 1}{v_i} = \sum_{s=1}^{r_i} \binom{r_i}{s} v_i^{s-1}.\end{aligned}$$

(f) This is a consequence of (e). □

2 Proofs

Construction — Use the notation of the construction above, and as in the Lemma, take

$$u = x^* - 1 \in \mathbb{Z}P^* = \mathbb{Z}(P/P').$$

Note that if $i \in \{2, 3, 4\}$ and $i + j \geqslant 5$, then $x_i = x_2^{u^{i-2}}$ and $x_i^{u^j} = 1$. Using Lemma (e), we get

$$\begin{aligned}1 &= [x_3, x^p] = [x_3, x]^p = [x_2, x]^{pu} = x_4^p, \\ 1 &= [x_2, x^p] = [x_2, x]^{p+p(p-1)u/2} = [x_2, x]^p = [x_1, x]^{pu} = x_3^p, \\ 1 &= [x_1, x^p] = [x_1, x]^{p+p(p-1)u/2+p(p-1)(p-2)u^2/6} = [x_1, x]^p = x_2^p,\end{aligned}$$

so $P_2^p = \langle x_2^p, x_3^p, x_4^p \rangle = 1$. As in Lemma (a), it follows that P' can be regarded as an $\mathbb{F}_p P^*$-module.

Now the relations (1) imply that P can be constructed as follows. Write $\mathbb{C}_n$ for the cyclic group of order n, and let $X_0 = \langle x_1, x_2 \rangle$ be a nonabelian group of order p^3 and exponent p^2, with $\langle x_1 \rangle \simeq \mathbb{C}_{p^2}$, $\langle x_2 \rangle \simeq \mathbb{C}_p$ and $x_1^{x_2} = x_1^{1-kp}$. Take $X_3 = \langle x_3 \rangle \simeq \mathbb{C}_p$, and form the direct product $M = X_0 \times X_3$. Put $x_4 = [x_2, x_1] = x_1^{kp}$, and $N = \langle x_2, x_3, x_4 \rangle$.

Next take $X = \langle x \rangle$, and make x act on the elements x_i by taking $x_i^x = x_i x_{i+1}$ $(i = 1, 2, 3)$. Using a well known relation in groups of class 2 [9, III (1.3.b)], we get

$$(x_1x_2)^{kp} = x_1^{kp}x_2^{kp}[x_2, x_1]^{kp(kp-1)/2} = x_1^{kp},$$
$$(x_1^x)^{p^2} = (x_1x_2)^{p^2} = 1,$$
$$(x_j^x)^p = (x_jx_{j+1})^p = 1 \quad (j = 2, 3),$$
$$[x_2^x, x_1^x] = [x_2x_3, x_1x_2] = [x_2, x_1] = x_1^{kp} = (x_1x_2)^{kp} = (x_1^x)^{kp},$$
$$[x_3^x, x_i^x] = [x_3x_4, x_ix_{i+1}] = 1 \quad (i = 1, 2).$$

This shows that x preserves the relations of M, so $x \in \operatorname{Aut} M$, and we form the corresponding semidirect product $L = XM$.

Then $L'' = N' = 1$, and N is an $\mathbb{F}_p(L/L')$-module, as in Lemma (a). Let $* : L \to L/L'$ be the natural homomorphism, and take $u = x^* - 1 \in \mathbb{F}_pL^*$. Now

$$x_4^x = (x_1x_2)^{kp} = x_1^{kp} = x_4,$$

and hence $x_{i+1}^{u^j} = 1$ when $i \in \{1, 2, 3\}$ and $j \geqslant 3$. Applying Lemma (e), we get

$$[x_i, x^p] = x_{i+1}^{p+p(p-1)u/2+p(p-1)(p-2)u^2/6} = 1 \quad (i = 1, 2, 3)$$

because $p \geqslant 5$. Thus $X \simeq \mathbb{C}_p$, so $P = L$, and therefore $|P| = p^5$.

Now suppose $3 \mid p - 1$ and let k_1 be a primitive cube root of 1 modulo p^2. Take $A = \langle a \rangle \simeq \mathbb{C}_3$, and make a act on x and x_1 by taking

$$x^a = x^{k_1}, \quad x_1^a = x_1^{k_1^2}.$$

Using the relations (1), together with Lemma (f) (and copying the calculation at the end of Remark (c) above), we get

$$(x_1^a)^{kp} = (x_1^{kp})^{k_1^2},$$
$$[x_1^a, x^a, x_1^a] = [x_1^{k_1^2}, x^{k_1}, x_1^{k_1^2}] = [x_1, x, x_1]^{k_1^2},$$
$$[x_1^a, x^a, x^a, x^a] = [x_1^{k_1^2}, x^{k_1}, x^{k_1}, x^h] = [x_1, x, x, x]^{k_1^2},$$

and hence $(x_1^a)^{kp} = [x_1^a, x^a, x_1^a] = [x_1^a, x^a, x^a, x^a]$. It is also clear that $(x^a)^p = (x_1^a)^{p^2} = [(x_1^a)^p, x^a] = 1$, so a preserves the relations (1). Thus $a \in \operatorname{Aut} P$, and we form the corresponding semidirect product $H = AP$, with

$$|H| = |A| \cdot |P| = 3p^5.$$

Next take the additive abelian groups

$$U = \langle x \rangle P'/P', \quad U_i = \langle x_i \rangle P_{i+1}/P_{i+1} \quad (i = 1, 2, 3, 4).$$

Let $W_i = \mathbb{F}_p w_i$ be a 1-dimensional vector space over $\mathbb{F}_p$, and make W_i a (right) $\mathbb{F}_p A$-module by taking

$$w_i a = k_1^i w_i \quad (i = 0, 1, 2).$$

Since $x_1^a = x_1^{k_1^2}$, we can use induction to show that

$$x_i^a = [x_{i-1}, x]^a \equiv [x_{i-1}^{k_1^i}, x^{k_1}] \equiv x_i^{k_1^{i+1}} \pmod{P_{i+1}} \quad (i = 2, 3, 4),$$

and hence

$$\left.\begin{array}{rcl} P/P' & = & U \oplus U_1 \simeq W_1 \oplus W_2 \\ P_i/P_{i+1} & = & U_i \simeq W_{i-2} \quad (i = 2, 3, 4) \end{array}\right\} \tag{2}$$

Finally put

$$G = \operatorname{Aut} H, \quad \Gamma = C_G(H/P').$$

Proof of the Theorem — It follows from (2) that

$$Z(H) = 1, \quad H = \operatorname{Inn} H \leqslant G,$$

where $\operatorname{Inn} H$ is the group of inner automorphisms of H. We next calculate the automorphism group G/Γ induced on H/P', and we claim that

$$G/\Gamma = \operatorname{Inn}(H/P') = H/P'. \tag{3}$$

To prove this, suppose $\theta \in G$. By Sylow's theorem, there is an element $g \in H$ such that $A^\theta = A^g$, and replacing θ by θg^{-1}, we may assume that $A^\theta = A$. Then θ acts on P/P', and permutes the set of $\mathbb{F}_p A$-submodules $\{U, U_1\}$; more precisely, either $a^\theta = a$ and θ stabilises U and U_1, or else $a^\theta = a^2$ and θ interchanges U and U_1. But $\langle x_1 \rangle P' = \Omega_1(P)$ is the subgroup generated by the elements of order p in P, which is a characteristic subgroup of P, so the first alternative must apply. Moreover there are integers r, s and elements $y, z \in P'$, such that $x^\theta = x^r y$ and $x_1^\theta = x_1^s z$. Then the relations (1) imply that $s \equiv rs^2 \equiv r^3 s \pmod p$, and hence $r^3 \equiv rs \equiv 1 \pmod p$ [1, page 303]. This means that θ acts on P/P' in the same way as an element of A, which completes the proof of (3).

It follows from (2) that $H \cap \Gamma = P'$, and using (3) we get

$$G/\Gamma = H/P' = H/H \cap \Gamma \simeq H\Gamma/\Gamma, \quad G = H\Gamma,$$
$$G/P' = H\Gamma/P' = (H/P') \times (\Gamma/P'), \quad G/H \simeq \Gamma/P'.$$

But $P' = \Phi(H)$ is the Frattini subgroup of H, and hence Γ is a p-group [9, III (3.18)], so G/H is a p-group. Also $P = [P, A]$, and therefore $H = \langle A^H \rangle = O^p(G)$. Now the Proposition implies that G is complete, and it remains to find $|G|$.

To do this, we investigate Γ. Since $AP' \lhd A\Gamma$, it follows from Frattini's argument [9, I (7.8)] that

$$A\Gamma = AP'N_{A\Gamma}(A) = AP'N_\Gamma(A) = AP'C_\Gamma(A),$$
$$\Gamma = P'C_\Gamma(A),$$

so it suffices to study $C_\Gamma(A)$. Now P' is elementary abelian, and Maschke's theorem [9, I (17.7)] implies that

$$P' = Z_0 \times P_3 = Z_0 \times Z_1 \times Z_2,$$

where Z_i is normalised by A, and is $\mathbb{F}_p A$-isomorphic to W_i $(i=0,1,2)$. Then $Z_0 = C_{P'}(A) \leqslant C_\Gamma(A)$. Moreover there is an automorphism $\gamma_1 \in C_\Gamma(A)$ defined by the equations

$$a^{\gamma_1} = a, \quad x^{\gamma_1} = x, \quad x_1^{\gamma_1} = x_1^{1+kp} = x_1 x_4,$$

with $\Gamma_1 = \langle \gamma_1 \rangle \simeq \mathbb{C}_p$ and $[P', \Gamma_1] = P' \cap \Gamma_1 = 1$. We claim that

$$C_\Gamma(A) = Z_0 \Gamma_1 \simeq \mathbb{C}_p \times \mathbb{C}_p. \tag{4}$$

Clearly $Z_0\Gamma_1 \leqslant C_\Gamma(A)$. To prove the opposite inclusion, suppose $\gamma \in C_\Gamma(A)$, and consider the map $\delta : P \to P'$ defined by taking

$$y^\delta = [y, \gamma] = y^{-1} y^\gamma \quad (y \in P).$$

If $y_1, y_2 \in P$ then

$$\begin{aligned} (y_1 y_2)^\delta &= [y_1 y_2, \gamma] = [y_1, \gamma]^{y_2} [y_2, \gamma] \\ &\equiv [y_1, \gamma][y_2, \gamma] = y_1^\delta y_2^\delta \pmod{P_3}, \end{aligned}$$

$$y_1^{a\delta} = [y_1^a, \gamma] = [y_1, \gamma]^a = y_1^{\delta a},$$

and hence δ induces an $\mathbb{F}_p A$-homomorphism $P/P' \to P'/P_3$. But (2) implies that $P/P' \simeq W_1 \oplus W_2$ and $P'/P_3 \simeq W_0$, so the only such homomorphism is the zero map, and therefore $P^\delta \leqslant P_3$.

Then the same calculation shows that δ induces an $\mathbb{F}_p A$-homomorphism $\delta_1 : P/P' \to P_3/P_4$. Now (2) implies that

$$P/P' = U \oplus U_1,$$

with

$$\mathrm{Hom}_{\mathbb{F}_p A}(U_1, P_3/P_4) = \mathrm{Hom}_{\mathbb{F}_p A}(W_2, W_1) = 0,$$

and therefore

$$\delta_1 \in \mathrm{Hom}_{\mathbb{F}_p A}(U, P_3/P_4) = \mathrm{End}_{\mathbb{F}_p A}\, W_1 = \mathbb{F}_p$$

[9, V (4.3)]. But it follows from (1) that $[x_1, x_2] = x_4^{-1} \in P_4$, and $[x, x_2] = x_3^{-1} \in x_3^{-1} P_4$. Hence there exist an integer r, and an element $z \in Z_0$, such that

$$(yP')^{\delta_1} = [y, \gamma]P_4 = [y, x_2^r]\, P_4 = [y, z]P_4 \quad (y \in P).$$

Replacing γ by γz^{-1}, we may assume that $P^\delta \leqslant P_4$.

Finally we now get $\delta \in \mathrm{Hom}_{\mathbb{F}_p A}(P/P', P_4)$, and a similar argument shows that there is an integer s such that

$$y^\delta = [y, \gamma] = [y, \gamma_1^s] \quad (y \in P).$$

This means that $\gamma = \gamma_1^s \in \Gamma_1$, which completes the proof of (4). Therefore

$$G = H\Gamma = HC_\Gamma(A) = H\Gamma_1, \quad |G| = |H| \cdot |\Gamma_1| = 3p^6. \qquad \square$$

Remark We can describe $Z_0 = C_{P'}(A)$ more explicitly as follows. As in Lemma (b), the elements of P' can be written as expressions of the form x_2^S, where S is in the group algebra $\mathbb{F}_p P^*$. As before, put

$$u = x^* - 1, \quad u_1 = x_1^* - 1,$$

and note that the equations (1) imply that the group algebra can be replaced by the quotient got by imposing the relations

$$u_1 = u^2, \quad u^3 = uu_1 = u_1^2 = 0. \qquad (5)$$

We seek generators z_i for the subgroups Z_i.

Since $x_4^a = x_4^{k_1^2}$, we can take $z_2 = x_4$, with $Z_2 = \langle z_2 \rangle$. Using (5) and Lemma (e), we get $x_3^a = [x_1^{k_1^2}, x^{k_1}, x^{k_1}] = x_3^S = x_2^{uS}$, where

$$S = k_1^2 \left(k_1 + \binom{k_1}{2} u\right)^2 = k_1^2 \left(k_1^2 + 2k_1 \binom{k_1}{2} u\right)$$
$$= k_1 + k_1(k_1 - 1)u,$$

so $x_3^a = x_3^{k_1} x_4^{k_1^2 - k_1}$. Put

$$z_1 = x_3 x_4^{-1}, \qquad \text{with} \quad \begin{cases} [z_1, x] = x_4 = z_2 \\ [z_1, x_1] = 1 \end{cases}$$

Then $z_1^a = x_3^{k_1} x_4^{k_1^2 - k_1} \cdot x_4^{-k_1^2} = x_3^{k_1} x_4^{-k_1} = z_1^{k_1}$, and therefore $Z_1 = \langle z_1 \rangle$.

Using (5) and Lemma (e) again, we get $x_2^a = [x_1^{k_1^2}, x^{k_1}] = x_2^T$, where

$$T = \left(k_1^2 + \binom{k_1^2}{2} u_1\right) \left(k_1 + \binom{k_1}{2} u + \binom{k_1}{3} u^2\right)$$
$$= 1 + \frac{k_1 - 1}{2} u + \frac{(k_1 - 1)(k_1 - 2)}{6} u^2 + \frac{k_1^2 - 1}{2} u_1$$
$$= 1 + \frac{k_1 - 1}{2} u + \frac{4k_1^2 - 3k_1 - 1}{6} u^2,$$

so $x_2^a = x_2 x_3^{(k_1-1)/2} x_4^{(4k_1^2 - 3k_1 - 1)/6}$. Put

$$z_0 = x_2 x_3^{-1/2} x_4^{-1/6}, \qquad \text{with} \quad \begin{cases} [z_0, x] = x_3 x_4^{-1/2} = z_1 z_2^{1/2} \\ [z_0, x_1] = x_4 = z_2 \end{cases}$$

Then

$$z_0^a = x_2 x_3^{(k_1-1)/2} x_4^{(4k_1^2 - 3k_1 - 1)/6} \cdot x_3^{-k_1/2} x_4^{-(k_1^2 - k_1)/2} \cdot x_4^{-k_1^2/6}$$
$$= x_2 x_3^{-1/2} x_4^{-1/6} = z_0,$$

and hence $Z_0 = \langle z_0 \rangle$.

Finally let γ_0 be the inner automorphism induced by z_0^{-1}. Then $C_\Gamma(A) = \langle \gamma_0, \gamma_1 \rangle$, with

$$x^{\gamma_0} = x[x, z_0^{-1}] = x z_1 z_2^{1/2},$$

$$x_1^{\gamma_0} = x_1[x_1, z_0^{-1}] = x_1 z_2.$$

Using a well known relation [9, III (1.3.b)], we get

$$(x^r)^{\gamma_0} = (x z_1 z_2^{1/2})^r = x^r z_1^r [z_1, x]^{r(r-1)/2} \cdot z_2^{r/2} = x^r z_1^r z_2^{r^2/2},$$
$$(x_1^r)^{\gamma_0} = (x_1 z_2)^r = x_1^r z_2^r.$$

These exponents correspond to the matrix entries in the (more elaborate) examples constructed by Heineken [6].

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