**Advances in Group Theory and Applications**
© 2016 AGTA - www.advgrouptheory.com/journal
2 (2016), pp. 67–77
ISSN: 2499-1287
DOI: 10.4399/97888548970145

# Seminormal, Non-Normal Maximal Subgroups and Soluble PST-Groups

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(*Received Apr. 9, 2016; Accepted Aug. 11, 2016 — Communicated by M.R. Dixon*)

**Abstract**

All groups in this paper are finite. Let G be a group. Maximal subgroups of G are used to establish several new characterisations of soluble PST-groups.

*Mathematics Subject Classification (2010)*: 20D20, 20D10, 20F16

*Keywords*: finite group; permutability; seminormality; S-semipermutability

## 1 Introduction and statement of results

All groups in this paper are finite.

There are many articles in the literature (for instance, [1],[5],[3],[6] to name just the four classical ones) where global information about a group G is obtained by assuming that some members of relevant families of subgroups of G are either normal or satisfy a sufficiently strongly embedding property extending normality. In many cases, the subgroups are the normal subgroups of G, and the embedding assumptions are that they are permutable or S-permutable in G.

Recall that a subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. H is said to be *permutable* (respectively, *S-permutable*) in G if H permutes with all subgroups (respectively, Sylow subgroups) of G. Examples of permutable subgroups include the normal subgroups of G. Non-Dedekind modular groups and non-modular nilpotent groups show that S-permutability, permutability and normality are quite different subgroup embedding properties. However, according to a result of Kegel [12], every S-permutable subgroup of G is always subnormal.

A group G is a PST-*group* if every subnormal subgroup of G is S-permutable in G. In the same way classes of PT-*groups* and T-*groups* are defined, in which every subnormal subgroup is permutable or normal respectively. Since normal subgroups are permutable and obviously permutable subgroups are S-permutable then it follows that T is a proper subclass of PT and PT is a proper subclass of PST. Soluble PST, PT and T-groups were studied and characterised by Agrawal [1], Zacher [15] and Gaschütz [10] respectively.

**Theorem 1**

1. *A soluble group* G *is a* PST-*group if and only if the nilpotent residual* L *of* G *is an abelian Hall subgroup of* G *on which* G *acts by conjugation as power automorphisms.*

2. *A soluble* PST-*group* G *is a* PT-*group (respectively* T-*group) if and only if* $G/L$ *is a modular (respectively Dedekind) group.*

Note that if G is a soluble T, PT or PST-group then every subgroup and every quotient of G inherits the same properties.

We mention that in [5, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and S-permutability are semipermutability and S-semipermutability introduced by Chen in [8]: a subgroup X of a group G is said to be *semipermutable* (respectively, *S-semipermutable*) in G provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|X|, |K|) = 1$. A semipermutable subgroup of a group need not be subnormal. For example a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (respectively, S-semipermutable) subgroup of a group G must be normalised by every subgroup (respectively, Sylow subgroup) $P$ of $G$ such that $\gcd(|X|,|P|)=1$. This observation was the basis for Beidleman and Ragland [7] to introduce the following subgroup embedding properties.

A subgroup $X$ of a group $G$ is said to be *seminormal* (respectively, *S-seminormal*)[1] in $G$ if it is normalised by every subgroup (respectively, Sylow subgroup) $K$ of $G$ such that $\gcd(|X|, |K|) = 1$.

By [7, Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal: it is enough to consider a non-subnormal subgroup $H$ of a group $G$ such that $\pi(H) = \pi(G)$. The following result is an interesting characterisation of soluble PST-groups.

**Theorem 2** ([7]) *Let $G$ be a soluble group. Then the following statements are pairwise equivalent:*

1. $G$ *is a* PST*-group.*
2. *All the subnormal subgroups of* $G$ *are seminormal in* $G$*.*
3. *All the subnormal subgroups of* $G$ *are semipermutable in* $G$*.*
4. *All the subnormal subgroups of* $G$ *are S-semipermutable in* $G$*.*

**Definition 3** Let $G$ be a group. Then

1. $G$ is called an $S(n)NM$-*group* if every non-seminormal subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.
2. $G$ is called an $S(s)NM$-*group* if every non-S-semipermutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.
3. $G$ is called a $P(s)NM$-*group* if every non-semipermutable subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

The following three theorems provide some new and different characterisations of soluble PST-groups.

[1] Note that the term *seminormal* has different meanings in the literature

**Theorem A** *A group* $G$ *is a soluble* PST*-group if and only if every subgroup of* $G$ *is an* $S(s)NM$*-group.*

**Theorem B** *A group* $G$ *is a soluble* PST*-group if and only if every subgroup of* $G$ *is an* $S(n)NM$*-group.*

**Theorem C** *A group* $G$ *is a soluble* PST*-group if and only if every subgroup of* $G$ *is a* $P(s)NM$*-group.*

Robinson [13] introduced classes of groups in which cyclic subnormal subgroups are S-permutable, permutable or normal.

**Definition 4** A group $G$ is called a $PST_c$*-group* if every cyclic subnormal subgroup of $G$ is S-permutable in $G$.

Similarly, classes $PT_c$ and $T_c$ are defined, by requiring cyclic subnormal subgroups to be permutable or normal respectively. Robinson [13] provided characterisations for both soluble and insoluble cases. Here we mention only the soluble case.

**Theorem 5** ([13]) *Let* $G$ *be a group and* $F = F(G)$*, the Fitting subgroup of* $G$*.*

1. $G$ *is a soluble* $PST_c$*-group if and only if there is a normal subgroup* $L$ *such that,*
   a) $L$ *is abelian and* $G/L$ *is nilpotent.*
   b) $p'$*-elements of* $G$ *induce power automorphisms in the Sylow* $p$*-subgroup* $L_p$ *of* $L$ *for all primes* $p$*.*
   c) $\pi(L) \cap \pi(F/L) = \emptyset$
2. *A soluble* $PST_c$*-group is supersoluble.*
3. *A soluble group* $G$ *is a* $PT_c$ $(T_c)$*-group if and only if* $G$ *is a soluble* $PST_c$*-group such that all the elements of* $G$ *induce power automorphisms in* $L$ *and* $F/L$ *is a modular (Dedekind) group, where* $L$ *is the subgroup described in 1.*

Note that the important distinction between soluble PST-groups and soluble $PST_c$-groups is that the nilpotent residual is a Hall subgroup of the Fitting subgroup whereas the nilpotent residual of a soluble PST-group is a Hall subgroup of the entire group. In fact, Robinson in [13] showed that the sets of primes $\pi(L)$ and $\pi(G/L)$ can have

a large intersection, even when G is a soluble $T_c$-group.
It is clear that a soluble $PST_c$-group such that the nilpotent residual is a Hall subgroup of G is a PST-group. Also, note that the class of all soluble $PST_c$-group is much different than the class of soluble PST-group as the following theorem shows.

**Theorem 6** ([13]) *Let* G *be a group. Then*

1. *If every subgroup of* G *is a* $PST_c$*-group, then* G *is a soluble* PST*-group.*
2. *If every quotient of* G *is a soluble* $PST_c$*-group, then* G *is a soluble* PST*-group.*

In addition, a $PST_c$-group is a $PT_c$ ($T_c$)-group if all of its Sylow subgroups are modular (Dedekind) respectively [13].

There are similar connections as in Theorems 2 and 5 with classes $PST_c$, $PT_c$ and $T_c$ as seen in the next two theorems.

**Theorem 7** ([4]) *Let* G *be a soluble group. Then the following statements are pairwise equivalent:*

1. G *is a* $PST_c$*-group.*
2. *All the cyclic subnormal subgroups of* G *are seminormal in* G*.*
3. *All the cyclic subnormal subgroups of* G *are semipermutable in* G*.*
4. *All the cyclic subnormal subgroups of* G *are S-semipermutable in* G*.*

**Theorem 8** ([4]) *Let* G *be a soluble group with abelian nilpotent residual* L*. Then:*

1. G *is a* $PT_c$ ($T_c$)*-group if and only if every cyclic subnormal subgroup of* G *is seminormal in* G*, all the elements of* G *induce power automorphisms in* L*, and* F/L *is a modular (Dedekind) group.*
2. G *is a* $PT_c$ ($T_c$)*-group if and only if every cyclic subnormal subgroup of* G *is semipermutable in* G*, all the elements of* G *induce power automorphisms in* L*, and* F/L *is a modular (Dedekind) group.*
3. G *is a* $PT_c$ ($T_c$)*-group if and only if every cyclic subnormal subgroup of* G *is S-semipermutable in* G*, all the elements of* G *induce power automorphisms in* L*, and* F/L *is a modular (Dedekind) group.*

4. $G$ *is a* $PT_c$ $(T_c)$*-group if and only if* $G$ *is an* $PST_c$*-group such that all the elements of* $G$ *induce power automorphisms in* $L$*, and* $F/L$ *is a modular (Dedekind) group.*

**Definition 9** Let G be a group.

1. G is called a $S(n)NM_c$-*group* if every cyclic non-seminormal subgroup of G is contained in a non-normal maximal subgroup of G.

2. G is called an $S(s)NM_c$-*group* if every cyclic non-S-semipermutable subgroup of G is contained in a non-normal maximal subgroup of G.

3. G is called a $P(s)NM_c$-*group* if every cyclic non-semipermutable subgroup of G is contained in a non-normal maximal subgroup of G.

We now list three theorems that are similar to Theorems A, B and C; however we only consider certain subgroups of a group which are contained in non-normal maximal subgroups.

**Theorem D** *Let* $G$ *be a group. Then*

1. *If every subgroup of* $G$ *is an* $S(n)NM_c$*-group, then* $G$ *is a soluble* $PST_c$*-group and so* $G$ *is a soluble* $PST$*-group.*

2. *If every subgroup of* $G$ *is a* $PST_c$*-group, then* $G$ *is an* $S(n)NM_c$*-group and hence a soluble* $PST$*-group.*

**Theorem E** *Let* $G$ *be a group. Then*

1. *If every subgroup of* $G$ *is an* $S(s)NM_c$*-group, then* $G$ *is a soluble* $PST_c$*-group and so* $G$ *is a soluble* $PST$*-group.*

2. *If every subgroup of* $G$ *is a* $PST_c$*-group, then* $G$ *is an* $S(s)NM_c$*-group and a soluble* $PST$*-group.*

**Theorem F** *Let* $G$ *be a group. Then*

1. *If every subgroup of* $G$ *is a* $P(s)NM_c$*-group, then* $G$ *is a soluble* $PST_c$*-group and so is a soluble* $PST$*-group.*

2. *If every subgroup of* $G$ *is a* $PST_c$*-group, then* $G$ *is a* $P(s)NM_c$*-group and a soluble* $PST$*-group.*

# 2 Preliminaries

The lemmas encountered here are used in the proofs of the main theorems of this paper.

**Lemma 10** ([5, Theorem 2.1.8, p. 57])

1. *Let* $G$ *be a soluble group and let* $L$ *be the nilpotent residual of* $G$. *Then* $G$ *is a* PST*-group if and only if* $L$ *is an abelian Hall subgroup of* $G$ *and* $G$ *acts by conjugation on* $L$ *as a group of power automorphisms.*

2. *A soluble group is a* PST*-group if and only if every subnormal subgroup of* $G$ *is S-permutable (seminormal, semipermutable in* $G$*).*

**Lemma 11** ([14, Theorem 13.3.7, p. 399]) *Let* $N$ *be a minimal normal subgroup of a group* $G$. *Then* $N$ *normalizes all the subnormal subgroups of* $G$.

**Lemma 12** ([9, Theorem 5.9, p. 238; 14, Theorem 9.2.9, p. 265]) *A finite soluble group is generated by its system normalizers.*

**Lemma 13** ([14, Theorem 9.2.7, p. 264]) *Let* $G$ *be a finite soluble group and let* $L$ *be the nilpotent residual of* $G$. *If* $L$ *is abelian and* $D$ *is a system normalizer of* $G$, *then* $G = L \rtimes D$, *that is,* $G$ *is a semidirect product of* $L$ *by* $D$.

**Lemma 14** ([2, Corollary 1.3.3, p. 9]) *Let the finite group* $G = AB$ *be the product of two subgroups* $A$ *and* $B$. *Then for each prime* $p$ *there exist Sylow* $p$*-groups* $A_0$ *of* $A$ *and* $B_0$ *of* $B$ *such that* $A_0B_0$ *is a Sylow* $p$*-subgroup of* $G$.

# 3 Proof of the theorems

Proof of Theorem A — Let $G$ be a group. Assume that $G$ is a soluble PST-group, let $L$ be the nilpotent residual of $G$, and let $D$ be a system normalizer of $G$. By Lemma 10 $L$ is an abelian Hall subgroup of $G$ and $G$ acts by conjugation on $L$ as a group of power automorphisms. Moreover, by Lemma 13 $G = L \rtimes D$, the semidirect product of $L$ by $D$. We prove that $G$ is an $S(s)NM$-group by induction on $|G|$. Let $A$ be a non-S-semipermutable subgroup of $G$. Then $L \neq 1$ and $A \cap L \lhd G$. Also $A/A \cap L$ is a non-S-semipermutable subgroup of $G/A \cap L$. Now $A \cap L = 1$, for otherwise, by induction, $A/A \cap L$ would be contained in a non-normal maximal subgroup $M/A \cap L$ of $G/A \cap L$. Then $M$ would be a non-normal maximal subgroup of $G$ containing $A$. This would mean that $G$ is an $S(s)NM$-group. Hence $A \cap L = 1$. Since $L$ and $D$ are Hall subgroups we may assume $A \leqslant D$. Let $M$ be a maximal subgroup of $G$ containing $D$. Assume that $M \lhd G$, then $D^g \leqslant M$ for all $g \in G$ and so $D^G \leqslant M$. But $D^G = G$ by Lemma 12 so that $M$ is non-normal. Thus $A \leqslant M$ and hence, $G$ is an $S(s)NM$-group. Now applying [5, 2.1.9] we have every subgroup $H$ of $G$ is a soluble PST-group. Hence, by the argument above $H$ is an $S(s)NM$-group.

Now assume that every subgroup of $G$ is an $S(s)NM$-group but $G$ is not a soluble PST-group. Let $G$ be the counterexample of least order. Then every proper subgroup of $G$ is a soluble PST-group. Thus every proper subgroup of $G$ is supersoluble and hence $G$ is a soluble group. Since $G$ is not a PST-group there is a subnormal subgroup $H$ which is not S-semipermutable in $G$. Let $M$ be a maximal normal subgroup of $G$ such that $H \leqslant M$. Now $G$ is an $S(s)NM$-group so there is a non-normal maximal subgroup $L$ of $G$ such that $H \leqslant L$. Note that $G = LM$ and both $L$ and $M$ are soluble PST-subgroups of $G$. There is a Sylow $p$-subgroup $P$ of $G$ such that the gcd $(p, |H|) = 1$ and $H$ does not permute with $P$.

By Lemma 14 there is a Sylow $p$-subgroup $A$ of $L$ and a Sylow $p$-subgroup $B$ of $M$ such that $AB$ is a subgroup of $G$ and $AB \in Syl_p(G)$. Note that $H$ permutes with $A$ and $B$ so $H$ permutes with $AB = Q$. There is an element $x \in G$ such that $P^x = Q$. The properties of $G$ as stated in the Theorem are inherited by quotients, so if $N$ is a minimal normal subgroup of $G$ contained in $M$, then $(HN)P/N = P(HN)/N$ is a subgroup of $G/N$. Hence $P$ permutes with $HN$.

If $(HN)P$ is a proper subgroup of $G$, then, by the hypothesis of the theorem, $HP = PH$, which is a contradiction. Hence, $G = (HN)P$. By Lemma 11 $N$ normalizes $H$ and so $H \lhd HN$. Since $G = HNP$, there is an element $a \in P$ and $b \in HN$ such that $x = ab$. Thus, $H^b = H$ or $H^{b^{-1}} = H$ and $H$ permutes with $P^b$ so $HP = PH$, a final contradiction. $\square$

Proof of Theorem B — First assume that $G$ is a soluble PST-group. As in the proof of Theorem A we prove that $G$ is an $S(n)NM$-group in the same way we showed that $G$ is an $S(s)NM$-group in the proof of Theorem A. As in that proof, we use the fact that every subgroup

of G is a soluble PST-group to prove that every subgroup of G is an $S(n)NM$-group.
Conversely, assume that every subgroup of G is an $S(n)NM$-group but G is not a soluble PST-group and let G be such a group of smallest order. As in the proof of Theorem A, G is soluble and by Lemma 10, Part 2, there is a subnormal subgroup H of G which is not seminormal in G. There is a normal maximal subgroup M of G and a maximal subgroup L of G such that $G = LM$ and $H \leqslant L \cap M$. Now L and M are soluble PST-groups so that L (respectively, M) contains a Sylow p-subgroup A (respectively, B) such that AB is a Sylow subgroup of G. (Note this proof follows that of the proof of Theorem A). There is a Sylow p-subgroup P of G which does not normalize H but H is normalized by AB. So there is an $x \in G$ such that $P^x = Q = AB$. As in the proof of Theorem A a minimal normal subgroup N of G normalizes H and P normalizes HN in G. Also $G = HNP$.
Then there is an element $a \in P$ and an element $b \in HN$ such that $x = ab$ and $H^{b^{-1}} = H$ is normalized by $P^b$. This is the final contradiction. □

Proof of Theorem C — To obtain a proof of Theorem C just replace S-semipermutable in the proof of Theorem A by semipermutable and we obtain the desired proof. □

Proof of Theorem D — Suppose that every subgroup of the group G is an $S(n)NM_c$-group but G is not a soluble $PST_c$-group and we assume G is a counterexample of least order to the result. Then G is not a soluble $PST_c$-group but every proper subgroup of G is a soluble $PST_c$-group. By Theorem 5 (2) every proper subgroup of G is supersoluble and hence G is soluble.
By Theorem 7 (2) there is a cyclic subnormal subgroup H which is not seminormal in G. Hence there is a Sylow p-subgroup P such that P does not normalize H. As in the proof of Theorem A there exists a normal maximal subgroup L of G and a non-normal maximal subgroup M of G such that $G = LM$ and $H \leqslant L \cap M$. Since L and M are $S(n)NM_c$-subgroups of G, it follows from Lemma 13 that there are Sylow p-subgroups A of L and B of M such that AB is a Sylow p-subgroup of G and both A and B normalize H. There is an element $x \in G$ such that $P^x = AB$. Let $Q = AB$.
Consider a minimal normal subgroup N of G with $N \leqslant L$. We now consider the quotient $G/N$ of G. Since the properties of G, as enunciated in the statement of the theorem, are inherited by quotients of $S(n)NM_c$, the minimality of G implies $HN/N$ in $G/N$ is normal-

ized by $PN/N$. Hence, $P$ normalizes $HN$.
Also by Lemma 11 $N$ normalizes $H$ in $HN$. If $HNP$ is a proper subgroup of $G$, then $P$ normalizes $H$, which is a contradiction. Thus $HNP = G$. Let $x = ab$ where $b \in HN$ and $a \in P$, then $H^b = H$ and $P^x = P^b$. Hence

$$H^{b^{-1}} = H \text{ and } \left(P^b\right)^{b^{-1}} = P$$

normalizes $H$, a final contradiction.
Hence, $G$ is a soluble $PST_c$-group. Now let $X$ be a subgroup of $G$. Then every subgroup of $X$ is an $S(n)NM_c$-group so that our proof can be applied to $X$ to show that $X$ is a soluble $PST_c$-group. By Theorem 6 (1) $G$ is a soluble $PST$-group. This completes the proof of part (1) of Theorem D.
If every subgroup of $G$ is a $PST_c$-group, then by Theorem 6 (1) $G$ is a soluble $PST$-group. To show that every subgroup of $G$ is an $S(n)NM_c$-group follows from the necessity part of the proof of Theorem A. □

Proof of Theorem E — In the proof of Theorem E replace in Theorem D seminormal subgroup with S-semipermutable subgroup. Also replace $S(n)NM_c$ by $S(s)NM_c$. □

Proof of Theorem F — In the proof of Theorem F replace in Theorem D seminormal subgroup with semipermutable subgroup. Also replace $S(n)NM_c$ by $P(s)NM_c$. □

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