Basic 3-Transpositions of the Symplectic Group 
\[ \text{Sp}(2n, 2) \] *

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Abstract

In this paper we aim to study maximal pairwise commuting sets of 3-transpositions (transvections) of the simple symplectic group \( \text{Sp}(2n, 2) \), and to construct designs from these sets. Any maximal set of pairwise 3-transpositions is called a basic set of transpositions. Let \( G = \text{Sp}(2n, 2) \). It is well-known that \( G \) is a 3-transposition group with the set \( D \), the conjugacy class consisting of its transvections, as the set of 3-transpositions. Let \( L \) be a set of basic transpositions in \( D \). We aim to give general descriptions of \( L \) and \( 1 - (\nu, k, \lambda) \) designs \( D = (\mathcal{P}, \mathcal{B}) \), with \( \mathcal{P} = D \) and \( \mathcal{B} = \{ L^g \mid g \in G \} \). The parameters \( k = |L| \), \( \lambda \) and further properties of \( D \) are determined. We also, as examples, apply the method to the symplectic simple groups \( \text{Sp}(6, 2) \), \( \text{Sp}(8, 2) \) and \( \text{Sp}(10, 2) \).

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1 Introduction

Let \( G \) be a finite group generated by a class \( D \) of involutions such that any pair of non-commuting elements of \( D \) generate a dihedral group of order 6. Then \( D \) is called a class of conjugate 3-transpositions and \( G \)

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a 3-transposition group. Note that if $a, b \in D$ are such that $ab \neq ba$ then $o(ab) = 3$. Fischer ([6]) in his original studies on these groups considers the maximal commuting sets of 3-transpositions and denotes any such set by $L$. The set $L$ is defined to be a basic set of transpositions. The width of $G$ is defined to be the size of $L$ and is denoted by $w_D(G)$. The normalizer $N_G(L)$, that is the stabilizer of $L$ under conjugation, plays an important role in his classification of 3-transposition groups.

Let $G = \text{Sp}(2n, 2)$. It is well-known that $G$ is a 3-transposition group, where the set $D$ of 3-transpositions is the conjugacy class of its transvections. In this paper we aim first to study the maximal pair-wise commuting sets of 3-transpositions (transvections) of $G$. Let $L$ be a set of basic transpositions in $D$. We aim to give general descriptions of $L$. Secondly we aim to construct $1-(v, k, \lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $|\mathcal{P}| = D$ and $\mathcal{B} = \{Lg | g \in G\}$. The parameters $k = |L|$, $\lambda$ and further properties of $\mathcal{D}$ are determined. We also, as examples, apply the method to the symplectic simple groups $\text{Sp}(6, 2)$, $\text{Sp}(8, 2)$ and $\text{Sp}(10, 2)$.

Recently in [13] we applied our method to the several 3-transposition groups, namely the Symmetric groups $S_n$ and Fischer groups $F_i$ for $i \in \{21, 22, 23, 24\}$. We must also add here that a good number of publications has been devoted to constructing designs and codes from finite simple groups. For example interested readers could be referred to [5][8][9][10][11][14][15] and [16].

2 Background, terminology and basic results

Our notation will be standard, and it is as in [2] and [12] for designs, and ATLAS [4] for groups, finite simple groups and their maximal subgroups. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a $t$-$(v, k, \lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, and every $t$ distinct points are together incident with precisely $\lambda$ blocks. The complement of $\mathcal{D}$ is the structure $\tilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \tilde{\mathcal{I}})$, where $\tilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The dual structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, P) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is symmetric if it has the same number of points and blocks, and self dual if it is isomorphic to its dual.
The symplectic group $\text{Sp}(2n, 2)$

The groups $G.H$, $G:H$, and $G \cdot H$ denote a general extension, a split extension and a non-split extension respectively. For a prime $p$, $p^n$ denotes the elementary abelian group of order $p^n$.

Let $G$ be a finite 3-transposition group generated by a class $D$ of conjugate 3-transpositions. Fischer in [6] proved the following main theorem.

**Theorem 1** Let $G$ be a finite 3-transposition group such that

(i) $O_2(G)$ and $O_3(G)$ lie in the centre of $G$,

(ii) $G' = G''$. 

Then $G/Z(G)$ is isomorphic to a group in one of the following families:

(a) $S_n$, the symmetric groups,

(b) $\text{Sp}(2n, 2)$, the symplectic groups over $\text{GF}(2)$,

(c) $O^\mu(2n, 2)$, $\mu \in \{1, -1\}$, the orthogonal groups over $\text{GF}(2)$,

(d) $\text{PSU}(n, 2)$, the projective special unitary groups over $\text{GF}(4)$,

(e) $O^{\mu, \pi}(n, 3)$, $\mu, \pi \in \{1, -1\}$, the orthogonal groups over $\text{GF}(3)$,

(f) $F_{22}, F_{23}$ and $F_{24}$. The first two groups are simple and the third one contains a simple subgroup of index 2.

Let $G$ be one of the groups in the above list, and $L$ be a set of basic transpositions in $D$. Let $S$ be a Sylow 2-subgroup of $G$ containing $L$. Then we can easily show that $L = D \cap S$ and that $N_G(L)$ contains $\langle L \rangle$. It is also well-known that $N_G(L)$ contains a Sylow 2-subgroup of $G$ and its action (by conjugation) on $L$ is at least 2-transitive. Furthermore from the Fischer’s work we have $C_G(L) = \langle L \rangle$ and that $N_G(L)/C_G(L)$ is

(i) $S_{|L|}$ or $A_{|L|}$ in cases (a) or (e),

(ii) $\text{GL}(n, 2)$ in the case (b),

(iii) $\text{PSL}([n/2], 4)$ in the case (d),

(iv) the holomorph of an elementary abelian 2-group of order $2^n$ in the case (c),

(v) the Mathieu groups $M_{2i}$, $i \in \{2, 3, 4\}$, in the case (f).
For \( d \in D \) we define \( D_d = C_D(d) \setminus \{d\} \) and \( A_d = D \setminus C_D(d) \). Then \( D_d \) is a conjugacy class of elements of the group generated by \( D_d \). This property allowed Fischer to use induction in order to prove his results on the classification of 3-transposition groups.

**Proposition 2** Assume that \( G \) is acting primitively by conjugation on \( D \) and \( w_D(G) \geq 2 \), then

(i) \( G \) is rank 3 on \( D \),

(ii) \( C_G(d) \) has three orbits \( \{d\}, D_d \) and \( A_d \) on \( D \),

(iii) \( \langle D_d \rangle \) is transitive on \( D_d \),

(iv) \( \langle C_D(d) \rangle \) is transitive on \( A_d \).

**Proof** — See [6] and [1]. \( \square \)

### 3 The symplectic group \( \text{Sp}(2n, 2) \)

Assume \( G = \text{Sp}(2n, 2) \), the symplectic group acting on a \( 2n \)-dimensional symplectic space \( V \) over \( F = GF(2) \). Let \( D \) be the set of all symplectic transvections of \( G \). There is a one-one correspondence between \( D \) and the nonzero elements of \( V \) and hence \( |D| = 2^{2n} - 1 \) with

\[
|G| = 2^{n^2} (2^2 - 1)(2^4 - 1) \ldots (2^{2n} - 1).
\]

Using the above identification, we can see that for \( d \in D \), \( C_G(d) \) is the affine subgroup of the form \( 2^{2n-1}:\text{Sp}(2n - 2, 2) \), see for example Mpono [17]. Furthermore, \( G \) acts primitively on \( D \) and \( C_G(d) \) has three orbits \( \{d\}, D_d \) and \( A_d \) on \( D \) with

\[
|D_d| = 2(2^{2n-2} - 1), \ |A_d| = 2^{2n-1},
\]

and for \( x \in A_d \) we have \( \{x, d, d^x\} \) as a hyperbolic line (see Aschbacher [1]).

Let \( L \) be a set of basic 3-transpositions in \( D \). We know that

\[
N_G(L) = \langle L \rangle : \text{GL}(n, 2),
\]

is a maximal parabolic subgroup of \( G \) (see for example Wilson [18]). In the following we study the structure of \( L \) and deduce that \( \dim(\langle L \rangle) = n(n + 1)/2 \) with \( |L| = 2^n - 1 \).
Proposition 3 Let \( L \) be a set of basic transpositions of \( G = \text{Sp}(2n, 2) \). If \( S \) is a Sylow 2-subgroup of \( G \) containing \( L \), then \( S = \langle L \rangle : T_n \) where \( T_n \) is a Sylow 2-subgroup of \( \text{GL}(n, 2) \). Furthermore, viewing \( \langle L \rangle \) as a vector space over \( \text{GF}(2) \), \( \dim(\langle L \rangle) = n(n+1)/2 \).

Proof — Since \( S \supseteq \langle L \rangle \), by Section 2 we have that \( N_G(L) \) contains \( S \) and hence \( S \) is a Sylow 2-subgroup of \( N_G(L) \). Since

\[ N_G(L) = \langle L \rangle : \text{GL}(n, 2), \]

we have \( S = \langle L \rangle : T_n \) where \( T_n \) is a Sylow 2-subgroup of \( \text{GL}(2n, 2) \). Furthermore since \( |S| = 2^{n^2} \), we must have

\[ 2^{n^2} = |\langle L \rangle| \times |T_n| = \langle L \rangle \times 2^{n(n-1)/2} \]

and hence \( |\langle L \rangle| = 2^{n^2-[n(n-1)/2]} = 2^{n(n+1)/2} \). Since \( \langle L \rangle \) is an elementary abelian 2-group, we have \( \dim(\langle L \rangle) = n(n+1)/2 \).

Remark 4 (i) Using [7], it can be shown that \( \langle L \rangle \) consists of the following \( 2n \times 2n \) matrices over \( \text{GF}(2) \)

\[
\begin{pmatrix}
I_n & 0_n \\
X & I_n
\end{pmatrix},
\]

where \( X \) runs over all \( n \times n \) symmetric matrices over \( \text{GF}(2) \).

(ii) As we have seen in Section 2, \( L = D \cap S \). That is \( L \) is the set of all transvections in \( S \). Let us denote by \( T_u \), for any \( 0 \neq u \in V \), the corresponding transvection. Then, setting

\[ H = u^\perp, \ V = \langle w \rangle \oplus H, \ w \notin H, \]

we have \( T_u(u) = u \) and \( T_u(w) = w + u \). For \( T_u \in L = D \cap S \), by part (i) we must have the following matrix form for \( T_u \)

\[
\begin{pmatrix}
I_n & 0_n \\
X_u & I_n
\end{pmatrix},
\]

where \( X_u \) runs over all \( n \times n \) symmetric matrices over \( \text{GF}(2) \) satisfying \( w_2 X_u = u_1 \) with \( u_2 = 0 \), where \( u = (u_1|u_2) \) and \( w = (w_1|w_2) \) written as row vectors.
Let \( B = \{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\} \) be a symplectic basis for \( V \) and let \( f : V \times V \to \text{GF}(2) \) be a non-singular bilinear form on \( V \) such that all elements of \( B \) are perpendicular to each other except that \( f(e_i, f_i) = 1 \) for \( i \in \{1, 2, \ldots, n\} \). Let \( W_n = \langle e_1, e_2, \ldots, e_n \rangle \).

**Lemma 5** \( D \cap S = L = \{T_u : u = (u_1|0), \ 0 \neq u_1 \in W_n\} \), and hence \( |L| = 2^n - 1 \).

**Proof** — Let \( 0 \neq u = (u_1|u_2) \in V \) with corresponding transvection \( T_u \in D \). Then by Remark 4 all the transvections in \( D \cap S \) we must have \( u_2 = 0 \) and the matrix form of \( T_u \) with respect to \( B \) is

\[
\begin{pmatrix}
I_n & 0_n \\
X_u & I_n
\end{pmatrix}.
\]

Let \( u_1 = \sum_{i=1}^{n} \lambda_i e_i, \lambda_i \in \{0, 1\} \). Then clearly we must have

\[
T_u(f_i) = \begin{cases} 
  f_i + u & \text{if } \lambda_i = 1 \\
  f_i & \text{if } \lambda_i = 0.
\end{cases}
\]

Thus all the transvections in \( L \) are of the form \( T_u \) with \( 0 \neq u = (u_1|0), \ u_1 \in W_n. \) Therefore \( |L| = |W_n| - 1 = 2^n - 1 \). \( \square \)

**Remark 6** Consider \( L \) when \( n = 3 \). Then by Lemma 5 we have \( |L| = 2^3 - 1 = 7 \). Here we have

\[
B = \langle e_1, e_2, e_3, f_1, f_2, f_3 \rangle, \ W = \langle e_1, e_2, e_3 \rangle.
\]

Furthermore

\[
L = \{T_{e_1}, T_{e_2}, T_{e_3}, T_{e_1+e_2}, T_{e_1+e_3}, T_{e_2+e_3}, T_{e_1+e_2+e_3}\}
\]

with the following corresponding matrices:

\[
T_{e_1} \sim \begin{pmatrix} I_3 & 0_3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ T_{e_2} \sim \begin{pmatrix} I_3 & 0_3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
T_{e_3} \sim \begin{pmatrix} I_3 & 0_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T_{e_1+e_2} \sim \begin{pmatrix} I_3 & 0_3 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The symplectic group $\text{Sp}(2n, 2)$

$$T_{e_1+e_3} \sim \begin{pmatrix} I_3 & 0_3 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T_{e_2+e_3} \sim \begin{pmatrix} I_3 & 0_3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$T_{e_1+e_2+e_3} \sim \begin{pmatrix} I_3 & 0_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$ 

Note that, for example,

$$T_{e_1+e_3}(f_1) = f_1 + e_1 + e_3, \quad T_{e_1+e_3}(f_3) = f_3 + e_1 + e_3,$$

$$T_{e_1+e_3}(f_2) = f_2,$$

and

$$T_{e_1+e_2+e_3}(f_1) = f_1 + e_1 + e_2 + e_3,$$

$$T_{e_1+e_2+e_3}(f_2) = f_2 + e_1 + e_2 + e_3,$$

$$T_{e_1+e_2+e_3}(f_3) = f_3 + e_1 + e_2 + e_3.$$

4 Designs from basic transpositions of $\text{Sp}(2n, 2)$

Let $G = \text{Sp}(2n, 2)$. As we have seen in previous sections, $G$ is a 3-transposition group with the set $D$, the conjugacy class consisting of its transvections, as the set of 3-transpositions. Let $L$ be a set of basic transpositions in $D$. In Section 3 (see Proposition 3 and Lemma 5) we gave a general descriptions of $L$. In this section we aim to construct $1 - (v, k, \lambda)$ designs $\mathcal{D} = (\mathcal{P}, \mathcal{B})$, with $\mathcal{P} = D$ and $\mathcal{B} = \{Lg \mid g \in G\}$. The parameters $k = |L|, \lambda$ and further properties of $\mathcal{D}$ will be determined. We also, as examples, apply the method to the symplectic simple groups $\text{Sp}(6, 2), \text{Sp}(8, 2)$ and $\text{Sp}(10, 2)$.

**Theorem 7** Let $G = \text{Sp}(2n, 2)$ with $D$ as its conjugacy class of transvections and $B = L$ a set of basic transpositions in $D$. Let $\mathcal{B} = \{B^g \mid g \in G\}$, $\mathcal{P} = D$. Then we have a $1 - (2^n - 1, 2^n - 1, \lambda)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ with $\prod_{i=1}^{n}(1 + 2^i)$ blocks where $\lambda = \prod_{i=1}^{n-1}(1 + 2^i)$. Furthermore, The
group $G$ acts as an automorphism group of $\mathcal{D}$, primitive both on points and blocks of $\mathcal{D}$.

**Proof** — Note for $d \in D$ we have $|D| = [G : C_G(d)]$. As seen in Section 3, $C_G(d)$ is the affine subgroup of the form

$$2^{2n-1}:Sp(2n-2,2)$$

and $|D| = 2^{2n} - 1$. If $k$ is the size of each block, then since $B = L$, we have $|B| = k = |L| = 2^n - 1$ by Lemma 5. Now using Proposition 3, we have

$$G_B = \{g \in G : B^g = B\} = N_G(L) = \langle L \rangle : GL(n,2) \simeq 2^{n(n+1)/2} : GL(n,2).$$

Hence,

$$b = [G : N_G(L)] = |Sp(2n,2)|/[2^{n(n+1)/2} \times |GL(n,2)|] = \prod_{i=1}^{n} (1 + 2^i)$$

is the number of distinct blocks.

Suppose that there are $\lambda$ blocks $B_i$ containing $d$. If $d'$ is another element of $D$, then $d' = d^g$ for some $g \in G$ and hence the $\lambda$ blocks $B_i^g$ contain $d'$. Therefore we have a $1-(v,k,\lambda)$ design $\mathcal{D}$ with $v = |D|$. Since $kb = \lambda v$, we deduce that

$$\lambda = kb/v = |L| \times b/|D|$$

$$= (2^n - 1) \times \prod_{i=1}^{n} (1 + 2^i)/(2^{2n} - 1)$$

$$= \prod_{i=1}^{n} (1 + 2^i)/(2^n + 1) = \prod_{i=1}^{n-1} (1 + 2^i).$$

The action of $G$ on points arises from the action of $G$ on $D$. Now $B = B^G$ implies that $G$ is transitive on $B$ with

$$G_B = \{g \in G : B^g = B\}$$

as the stabiliser of the action on blocks. Clearly $G$ acts as an automorphism group on $\mathcal{D}$, primitive both on points and blocks. Since $G$ acts primitively on $D$ (note $C_G(d)$ is maximal in $G$), $G$ acts primitively on
The symplectic group \( \text{Sp}(2n, 2) \) has \( 2^n(n+1)/2 \times \text{GL}(n, 2) \) points of \( \mathcal{D} \). The action of \( G \) on \( \mathcal{B} \) is equivalent to the action of \( G \) on the cosets of \( G \)\( \mathcal{B} = N_G(L) \). Since \( N_G(L) = 2^n(n+1)/2 \times \text{GL}(n, 2) \) is maximal in \( G \) (a maximal parabolic subgroup), the action on blocks is also primitive. \( \square \)

**Corollary 8** Let \( \mathcal{D}_{2n} \) and \( \mathcal{D}_{2n-2} \) be designs constructed from basic transpositions of \( \text{Sp}(2n, 2) \) and \( \text{Sp}(2n-2, 2) \) respectively. Then

(i) \( \mathcal{D}_{2n} \) is a \( 1-(2^{2n} - 1, 2^n - 1, \lambda_{2n}) \) design with \( b_{2n} \) blocks, where

\[
\lambda_{2n} = \frac{b_{2n}}{(2^n + 1)},
\]

and

(ii) \( b_{2n} = (1 + 2^n) \times b_{2n-2} \) and \( \lambda_{2n} = (1 + 2^{n-1}) \times \lambda_{2n-2} = b_{2n-2} \).

**Proof** — (i) By Theorem 7, we have

\[
\lambda_{2n} = \prod_{i=1}^{n} \frac{(1 + 2^i)}{(2^n + 1)} = \frac{b_{2n}}{(2^n + 1)}.
\]

(ii) We have

\[
b_{2n} = \prod_{i=1}^{n} (1 + 2^i) = (1 + 2^n) \times \prod_{i=1}^{n-1} (1 + 2^i) = (1 + 2^n) \times b_{2n-2}.
\]

Now by part (i) we have \( \lambda_{2n} = b_{2n}/(2^n + 1) \), and hence \( \lambda_{2n} = b_{2n-2} \). Since

\[
\lambda_{2n-2} = \frac{b_{2n-2}}{(2^{n-1} + 1)} \quad \text{and} \quad \lambda_{2n} = b_{2n-2},
\]

we have \( \lambda_{2n-2} = \lambda_{2n}/(2^{n-1} + 1) \), i.e. \( \lambda_{2n} = \lambda_{2n-2} \times (2^{n-1} + 1) \). \( \square \)

We apply the results obtained in Sections 3 and 4 to \( \text{Sp}(6, 2), \text{Sp}(8, 2) \) and \( \text{Sp}(10, 2) \). These are summarized in Table 1. Computations with Magma [3] show that \(|\text{Aut}(\mathcal{D})| = |G|\), and since \( \text{Aut}(\mathcal{D}) \supseteq G \), we must have \( \text{Aut}(\mathcal{D}) = G \).
Table 1: Results for $\text{Sp}(6, 2)$, $\text{Sp}(8, 2)$, $\text{Sp}(10, 2)$

| $G$   | $|D|$ | $|L|$ | $(L)$ | $N_G(L)$ | $\mathcal{D}_{2n}$ | $\text{Aut}(\mathcal{D}_{2n})$ |
|-------|------|------|-------|---------|-------------------|-------------------------|
| $\text{Sp}(6, 2)$ | 63   | 7    | $2^6$ | $2^6:\text{GL}(3, 2)$ | $1 - (63, 7, 15)$ \hspace{1em} $\lambda_6 = 15$ \hspace{1em} $b_6 = 135$ | $\text{Sp}(6, 2)$ |
| $\text{Sp}(8, 2)$ | 255  | 15   | $2^{10}$ | $2^{10}:\text{GL}(4, 2)$ | $1 - (255, 15, 135)$ \hspace{1em} $\lambda_8 = 135$ \hspace{1em} $b_8 = 2295$ | $\text{Sp}(8, 2)$ |
| $\text{Sp}(10, 2)$ | 1023 | 31   | $2^{15}$ | $2^{15}:\text{GL}(5, 2)$ | $1 - (1023, 31, 2295)$ \hspace{1em} $\lambda_{10} = 2295$ \hspace{1em} $b_{10} = 75735$ | $\text{Sp}(10, 2)$ |

REFERENCES


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