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Groups whose Proper Subgroups of Infinite Rank are Minimax-by-Nilpotent or Nilpotent-by-Minimax [*]

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Abstract

Let $\mathfrak{M}$ denote the class of of soluble-by-finite minimax groups, and $\mathfrak{N}$ the class of nilpotent groups. The main result states that if G is a group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{MN}$-groups, then G is either in $\mathfrak{MN}$ or it is a group of Heineken-Mohamed type, provided that G satisfies a suitable generalized solubility condition. Moreover, we prove that a group of infinite rank all of its proper subgroups of infinite rank are $\mathfrak{NM}$-groups is itself a $\mathfrak{NM}$-group.

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1 Introduction

A group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. If there is no such r, the group G has infinite rank. In recent years, generalized soluble groups G of infinite rank whose proper subgroups of infinite rank belong to a given class $\mathfrak{Y}$ have been studied and it was proved that all proper subgroups of G are in the class $\mathfrak{Y}$, sometimes these groups G

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themselves belong to $\mathfrak{Y}$. The motivation of establishing such results arises from the classic theorems of Yu.I. Merzljakov [17] and V.P. Šunkov [23] concerning groups with rank restrictions on their abelian subgroups. We recall that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite image. In [6], N.S. Černikov considered the class $\mathfrak{X}$ obtained by taking the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$, $\mathbf{L}$. Clearly $\mathfrak{X}$ is a subclass of the class of locally graded groups. In his paper [6], Černikov proved that an $\mathfrak{X}$-group of finite rank is (locally soluble)-by-finite. The $\mathfrak{X}$-groups form a large $\mathbf{S}$-closed class of generalized soluble groups containing, in particular, the classes of locally (soluble-by-finite) groups, radical groups and residually finite groups. A result of B. Bruno and R.E. Phillips [4] shows that if G is an $\mathfrak{X}$-group of infinite rank in which all proper subgroups of infinite rank have finite commutator subgroup, then also the commutator subgroup G' of G is finite. Moreover, in [12], it is proved that a locally (soluble-by-finite) group G of infinite rank whose proper subgroups of infinite rank are abelian-by-finite, then all it's proper subgroups are abelian-by-finite. The aim of the present paper is to provide a further contribution to the topic, replacing the terms "abelian group" by "nilpotent group" and "finite group" by "$\mathfrak{Y}$-group", where $\mathfrak{Y}$ is a subclass of the class of soluble-by-finite minimax groups with certain conditions of closure operations. As we will see, $\mathfrak{Y}$ can be chosen to be the class $\mathfrak{C}$,$\mathfrak{PF}$ or $\mathfrak{M}$ of Chernikov, polycyclic-by-finite or soluble-by-finite minimax groups, respectively.

Recall that a group G is *minimal non-$\mathfrak{Y}$*, or briefly, MN$\mathfrak{Y}$, if it is not a $\mathfrak{Y}$-group, but all its proper subgroups belong to $\mathfrak{Y}$. The description of the structure of MN$\mathfrak{Y}$-groups can be considered as a first step in the investigation of groups of infinite rank whose proper subgroups of infinite rank belong to $\mathfrak{Y}$.

Most of our notation is standard and can be found in [21].

2 The minimax-by-nilpotent case

In this section we are interested in the classes $\mathfrak{FN}$, $\mathfrak{CN}$, $(\mathfrak{PF})\mathfrak{N}$, $\mathfrak{MN}$. We shall give the complete characterization of $\mathfrak{X}$-groups of infinite rank in which all proper subgroups of infinite rank are in these classes. The knowledge of the structure of MN$\mathfrak{FN}$-groups (and

MN$\mathfrak{C}\mathfrak{N}$-groups, MN$(\mathfrak{P}\mathfrak{F})\mathfrak{N}$-groups, MN$\mathfrak{M}\mathfrak{N}$-groups) will be relevant in our considerations.

We record the following obvious result, which will be frequently used. Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be two group classes, we say that $\mathfrak{Y}$ is $\mathfrak{Z}$-*characteristic* (a generalization of the idea of [13]) if any $\mathfrak{Y}\mathfrak{Z}$-group G contains a characteristic $\mathfrak{Y}$-subgroup A such that G/A is a $\mathfrak{Z}$-group. Note that if $\mathfrak{Y}$ is any subgroup closed class, then $\mathfrak{Y}$ is $\mathfrak{N}$-characteristic.

Lemma 2.1 *Let $\mathfrak{Y}$ and $\mathfrak{Z}$ be two group classes such that $\mathfrak{Y}$ is $\mathfrak{Z}$-characteristic and $\mathbf{N}_0$-closed and $\mathfrak{Z}$ is $\{\mathbf{H},\mathbf{N}_0\}$-closed. Then $\mathfrak{Y}\mathfrak{Z}$ is also $\mathbf{N}_0$-closed. Therefore if G is a MN$\mathfrak{Y}\mathfrak{Z}$-group, then every nilpotent image of G is a (possibly trivial) locally cyclic p-group for some prime p.*

Proof — Let H and K be normal $\mathfrak{Y}\mathfrak{Z}$-subgroups of a group G. Then there exists two subgroups A and B such that A (resp. B) characteristic in H (resp. K), A, B are $\mathfrak{Y}$-groups and H/A, K/B are $\mathfrak{Z}$-groups. Clearly AB is a normal $\mathfrak{Y}$-subgroup of G by the hypothesis. We also have that HK/AB is a $\mathfrak{Z}$-group, as it is the product of HB/AB and KA/AB. Hence HK is a $\mathfrak{Y}\mathfrak{Z}$-group. The rest of the claim follows by [19, Theorem 2.12]. □

Recall that a group G is of *Heineken-Mohamed type* [15], if G is infinite non-nilpotent with nilpotent and subnormal proper subgroups. The following result describe the structure of locally graded MN$\mathfrak{M}\mathfrak{N}$-groups. It shows in particular that such groups are precisely groups of Heineken-Mohamed type.

Theorem 2.2 *A group G is a locally graded MN$\mathfrak{M}\mathfrak{N}$ if and only if it is a group of Heineken-Mohamed type.*

Proof — Let G be a group as stated. Then, G cannot be finitely generated. As finitely generated $\mathfrak{M}\mathfrak{N}$-groups are obviously in $\mathfrak{M}$, it follows from the hypothesis that every finitely generated subgroup of G is an $\mathfrak{M}$-group. So G is locally in $\mathfrak{M}$ and, by [9, Theorem], either G is a finite rank-by-nilpotent group or G/G' is p-quasicyclic for some prime p and every proper subgroup of G is nilpotent. Assume for a contradiction that G is finite rank-by-nilpotent. As every nilpotent image of G is locally cyclic p-group for some prime p by Lemma 2.1, it thus follows that G is of finite rank.

Let us suppose first that G contains a proper normal subgroup N of finite index. N belongs to $\mathfrak{M}\mathfrak{N}$ and $\gamma_c(N)$ is a $\mathfrak{M}$-group for some positive integer c; there is no loss of generality if we assume that N

is nilpotent. If G is periodic then all its proper subgroups are Chernikov-by-nilpotent so, by [2, Corollary 2.1], G belongs to $\mathfrak{M}\mathfrak{N}$. Therefore G is non-periodic, so that N is non-periodic. An application of [22, 5.2.6] shows that also the factor group N/N' is non-periodic. Let M/N' be a maximal free abelian subgroup of N/N'; then

$$(N/N')/(M/N')$$

is periodic. Since G/N' has finite rank and M/N' has a finite number of conjugates in G/N', $(M/N')^G$ is a finitely generated abelian group. Furthermore, [2, Corollary 2.1] shows that the periodic group

$$(G/N')/(M/N')^G$$

is Chernikov-by-nilpotent hence, since every nilpotent image of G is Chernikov, $(G/N')/(M/N')^G$ is a Chernikov group. It follows that G/N' belongs to $\mathfrak{M}$, and so N/N' is an $\mathfrak{M}$-group. Therefore N belongs to $\mathfrak{M}$, as N is nilpotent. But this contradicts our assumption.

Thus G has no proper subgroup of finite index. By [6], G is locally soluble, and hence it has an ascending normal abelian series, so G is hyperabelian. By [21, Theorem 9.31], the finite residual of G is radicable nilpotent, and hence G is nilpotent, since it has no proper subgroup of finite index. This contradiction gives the result. □

Now we turn to the study of groups of infinite rank in which all proper subgroups of infinite rank belong to $\mathfrak{Y}\mathfrak{N}$, where $\mathfrak{Y}$ is an **S**-closed subclass of the class of soluble-by-finite minimax groups.

Theorem 2.3 *Let $\mathfrak{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is* **S***-closed. If G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are $\mathfrak{Y}\mathfrak{N}$-groups, then all proper subgroups of G are $\mathfrak{Y}\mathfrak{N}$-groups.*

Proof — Clearly all proper subgroups of G are finite rank-by-nilpotent. Using [9, Theorem], either G is a finite rank-by-nilpotent group or G/G' is p-quasicyclic for some prime p and every proper subgroup of G is nilpotent. In the latter case the result is immediate, so we may assume that G is finite rank-by-nilpotent. Then G contains a proper normal subgroup N such that G/N is nilpotent. Hence G/G' is of infinite rank, and so G contains a proper subgroup M of infinite rank such that $G' \leqslant M$ and G/M has infinite rank. Let H be any subgroup of G of finite rank. The product HM is a proper subgroup of infinite rank of G, and hence belongs to $\mathfrak{YN}$, so H also is a $\mathfrak{YN}$-group. Therefore all proper subgroups of G are $\mathfrak{YN}$-groups, as desired. □

Next, we derive some results on $\mathfrak{X}$-groups of infinite rank in which all proper subgroups of infinite rank are $\mathfrak{FN}$ (resp. $\mathfrak{CN}$, $(\mathfrak{PF})\mathfrak{N}$, $\mathfrak{MN}$).

The class $\mathfrak{F}$ of finite groups satisfies the hypotheses of Theorem 2.3; moreover, MN$\mathfrak{FN}$ of infinite rank are groups of Heineken-Mohamed type by [24, Theorem 3.5] and [4, Theorem 2]. The following result is valid.

Corollary 2.4 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $\mathfrak{FN}$*, then* G *is either in* $\mathfrak{FN}$ *or it is a group of Heineken-Mohamed type.*

Since the class $\mathfrak{PF}$ of polycyclic-by-finite groups satisfies the hypotheses of Theorem 2.3 so by [11, Theorem 3.9] and Corollary 2.4, we have the following consequence.

Corollary 2.5 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $(\mathfrak{PF})\mathfrak{N}$*, then* G *is either in* $(\mathfrak{PF})\mathfrak{N}$ *or it is a group of Heineken-Mohamed type.*

The class $\mathfrak{C}$ of Chernikov groups satisfies the hypotheses of Theorem 2.3 which, together with [2] enables us to deduce the following consequence.

Corollary 2.6 *Let* G *be* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $\mathfrak{CN}$*, then* G *is either in* $\mathfrak{CN}$ *or it is a group of Heineken-Mohamed type.*

The class $\mathfrak{M}$ of soluble-by-finite minimax groups satisfies the hypotheses of Theorem 2.3, 2.2 so we have the following consequence.

Corollary 2.7 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $\mathfrak{MN}$*, then* G *is either in* $\mathfrak{MN}$ *or it is a group of Heineken-Mohamed type.*

3 The nilpotent-by-minimax case

In this section, we consider the "dual" situation, that is the case in which proper subgroups of infinite rank of an $\mathfrak{X}$-group of infinite rank are $\mathfrak{NC}$ (resp. $\mathfrak{N}(\mathfrak{PF})$, $\mathfrak{NM}$). The results obtained should be seen in relation with the structure of MN$\mathfrak{NC}$-groups (resp. MN$\mathfrak{N}(\mathfrak{PF})$-groups, MN$\mathfrak{NM}$-groups).

In [18, 3], it is shown that a locally graded group whose proper subgroups are in $\mathfrak{NC}$ is itself in $\mathfrak{NC}$. As a continuation of that work, we give the soluble-by-finite minimax $\mathfrak{M}$ version by studying groups whose proper subgroups are $\mathfrak{NM}$. The result will be accomplished by a series of lemmas.

Lemma 3.1 *Let $\mathfrak{Y}$ be a class of groups contained in the class of finite rank groups which is* **H***-closed. If* G *is a* $\mathfrak{NY}$*-group, then* G *has a characteristic nilpotent subgroup* N *such that* G/N *is a* $\mathfrak{Y}$*-group.*

Proof — Let M be a normal nilpotent subgroup of G such that G/M is a $\mathfrak{Y}$-group. Consider the characteristic closure of M in G, and we write

$$\overline{M} := \langle \sigma(M), \sigma \in \mathrm{Aut}(G) \rangle .$$

Then $\overline{M}$ is characteristic in G and $G/\overline{M}$ is a $\mathfrak{Y}$-group. Since $\sigma(M) \lhd G$ and $\sigma(M) \simeq M$ for all $\sigma \in \mathrm{Aut}(G)$, $\overline{M}$ is generated by normal nilpotent subgroups and hence it locally nilpotent. Following the same way of the proof of [8, Lemma 1], we get that $\overline{M}$ is nilpotent. □

Lemma 3.2 *Let $\mathfrak{Y}$ be a class of groups contained in the class of finite rank groups and* $\{\mathbf{H}, \mathbf{P}\}$*-closed. If* G *is a group containing a normal* $\mathfrak{NY}$*-subgroup* N *such that* G/N *is in* $\mathfrak{Y}$*, then* G *is in* $\mathfrak{NY}$*.*

Proof — By Lemma 3.1, N contains a characteristic nilpotent subgroup L such that N/L is a $\mathfrak{Y}$-group. Since G/N and N/L are $\mathfrak{Y}$-groups and the class $\mathfrak{Y}$ is **P**-closed, we deduce that G/L belongs to $\mathfrak{Y}$ and so G is a $\mathfrak{NY}$-group. □

Corollary 3.3 *Let* G *be a group whose proper normal subgroups belong to* $\mathfrak{NM}$*. If* G *is imperfect, then* G *itself belongs to* $\mathfrak{NM}$*.*

Proof — If the quotient group G/G' is decomposable, then $G = MN$ is a product of two proper normal subgroups M and N. Since M and N are $\mathfrak{NM}$-groups, G is also a $\mathfrak{NM}$-group, by Lemma 3.1 and Lemma 2.1. Now let G/G' be an indecomposable group. Then G/G' is Chernikov by [19, Lemma 2.9] and so G lies in $\mathfrak{NM}$ by Lemma 3.2. □

Lemma 3.4 *Let* G *be a locally nilpotent group whose proper subgroups are in* $\mathfrak{NM}$*. Then* $G \in \mathfrak{NM}$*.*

Proof — Assume for a contradiction that G is a $\mathrm{MN}\mathfrak{NM}$-group. It follows from Corollary 3.3 and Lemma 3.2 that G is perfect and has no non-trivial finite images. Since the class of soluble groups is countably recognizable, G is countable. Let T be the torsion subgroup of G. If $G = T$ then, by [18, 3], G lies in $\mathfrak{NC}$, a contradiction. Hence $G \neq T$, and the factor group G/T is countable locally nilpotent torsion-free. Since G/T is the isolator of a proper subgroup K/T, G/T itself is soluble by [14, Lemma 4.6]. This contradiction establishes the result. □

It is now easy to prove the following result.

Theorem 3.5 *Let G be a locally graded group whose proper subgroups are in $\mathfrak{NM}$. Then $G \in \mathfrak{NM}$.*

Proof — Assume for a contradiction that the statement is false. In particular, G has no non-trivial finite images, see Lemma 3.2, and so G is locally (soluble-by-finite). Moreover, we deduce from Corollary 3.3 and Lemma 3.4 that G is perfect non-locally nilpotent. Let V be the Hirsch-Plotkin radical of G. Clearly, the factor group G/V is a $\mathrm{MN}\mathfrak{NM}$-group. Replacing G by G/V, it can be assumed without loss of generality that all ascendant subgroups of G are of finite rank. It is easy to see, using [8, Theorem 2], that if G has no simple images, then G is radicable nilpotent. By this contradiction we have that G contains a normal subgroup M such that $\overline{G} = G/M$ is an infinite simple group. Since the class of (locally soluble)-by-finite groups is countably recognizable by [7, Lemma 3.5], and $\overline{G}$ has no non-trivial finite images, we deduce that $\overline{G}$ is countable. Let $\overline{R}$ be the super-inert subgroup of $\overline{G}$, as described in [10, Proposition 1]. Since $\overline{G}$ is not locally finite (see [18, 3]), the Hirsch-Plotkin radical of $\overline{R}$ is trivial (see [10, Theorem 2]). Hence $\overline{R}$ is finite and then, as $\overline{G}$ is periodic over $\overline{R}$,$\overline{G}$ is periodic. This implies the contradiction that $\overline{G}$ is locally finite. □

The last part of this section is devoted to the proof of results concerning groups whose proper subgroups of infinite rank $\mathfrak{NC}$ (resp. $\mathfrak{N}(\mathfrak{PF})$, $\mathfrak{NM}$).

Lemma 3.6 *If G is a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are hypercentral-by-$\mathfrak{M}$, then G is not simple.*

Proof — Assume for a contradiction that G is simple. Note that the proper subgroups of G are either locally (soluble-by-finite) of finite rank or hypercentral-by-$\mathfrak{M}$. Since G is locally (soluble-by-finite), we deduce that the hypercentral-by-$\mathfrak{M}$ subgroups are (locally soluble)-by-finite. Thus all proper subgroups of G are (locally soluble)-by-finite and, since the class (locally soluble)-by-finite is countably recognizable by [7, Lemma 3.5]. It follows that G is countable, as G has no finite images. Let R be the super-inert subgroup of G, as described in [10, Proposition 1]. If R has a non-trivial normal locally nilpotent subgroup then its Hirsch-Plotkin radical is non-trivial, and if R has finite rank then its Hirsch-Plotkin radical is also non-trivial by [21, Lemma 10.39]. We deduce, using [10, Theorem 2], that G is locally finite. By [16], G is isomorphic to either $PSL(2, F)$ or $Sz(F)$ for some infinite locally finite field F. But each of these groups has a proper non hypercentral-by-Chernikov subgroup of infinite rank [20], a contradiction. Therefore the group G is not simple. □

Theorem 3.7 *Let $\mathfrak{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is $\{\mathbf{P}, \mathbf{S}, \mathbf{R}_0\}$-closed. If G is a perfect $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are in $\mathfrak{N}\mathfrak{Y}$, then all proper subgroups of G are in $\mathfrak{N}\mathfrak{Y}$.*

Proof — Assume for a contradiction that the statement is false. Let N be a proper subgroup of finite index in G. So there is a normal nilpotent subgroup L of N such that N/L is a $\mathfrak{Y}$-group. Since L has only finitely many conjugates in G and $\mathfrak{Y}$ is $\mathbf{R}_0$-closed, N/L_G is a $\mathfrak{Y}$-group, where, of course, L_G is the core of L in G. But then, as L_G is nilpotent and G/N belongs to $\mathfrak{Y}$, G is a $\mathfrak{N}\mathfrak{Y}$-group. Hence G has no non-trivial finite images and so it cannot be finitely generated. Now by [6], all proper subgroups of G are (locally soluble)-by-finite; in particular G is locally (soluble-by-finite).

Let N be a proper normal subgroup of infinite rank of G. If G/N has finite rank, then, in particular, all its proper subgroups are $\mathfrak{N}\mathfrak{M}$-groups, and hence G/N itself a $\mathfrak{N}\mathfrak{M}$-group by Theorem 3.5. Then G/N is soluble, a contradiction, because G is perfect. Thus G/N has infinite rank, so that if H is any proper subgroup of finite rank of G, then HN is a proper subgroup of infinite rank. It follows that HN belongs to $\mathfrak{N}\mathfrak{Y}$, and so H is a $\mathfrak{N}\mathfrak{Y}$-group. Therefore all proper normal subgroups of G are of finite rank. Suppose that G contains a normal subgroup M such that $\overline{G} = G/M$ is simple. But Lemma 3.6 shows that $\overline{G}$ cannot be simple. Hence G is of finite rank by [8, Theorem 2]. This contradiction completes the proof of the theorem. □

Now, by Theorem 3.7 and Theorem 3.5 , we have the following corollary.

Corollary 3.8 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $\mathfrak{N}\mathfrak{M}$*, then* G *itself is in* $\mathfrak{N}\mathfrak{M}$*.*

Proof — Clearly we may suppose that G is infinitely generated, see Lemma 3.2. So G is locally (soluble-by-finite) and using [1, Lemma 3.1], we conclude that all proper normal subgroups are in $\mathfrak{N}\mathfrak{M}$. By Corollary 3.3, one can assume that G is perfect. Application of Theorem 3.7 yields that all proper subgroups of G belong to $\mathfrak{N}\mathfrak{M}$. So G also is in $\mathfrak{N}\mathfrak{M}$ by Theorem 3.5. □

Following a similar approach using Theorem 3.7 and [18, 3], we deduce the following result.

Corollary 3.9 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are in* $\mathfrak{N}\mathfrak{C}$*, then* G *itself is in* $\mathfrak{N}\mathfrak{C}$*.*

Now we consider non-perfect groups of infinite rank whose proper subgroups of infinite rank are in $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$.

Lemma 3.10 *If* G *is a non-perfect group of infinite rank whose proper subgroups of infinite rank belong to* $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$*. Then either* $G \in \mathfrak{N}(\mathfrak{P}\mathfrak{F})$ *or* G/G' *is quasicyclic and* G' *is nilpotent.*

Proof — Suppose that G does not belong to $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$. In particular, G has no non-trivial finite images and so it is locally (soluble-by-finite). By [1, Lemma 3.1], all proper normal subgroups of G are in $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$. If the factor group G/G' is decomposable, then $G = MN$ is a product of two proper normal subgroups M and N. It follows from Lemma 3.1 and Lemma 2.1 that G is in $\mathfrak{N}\mathfrak{M}$. Thus G/G' is indecomposable, and hence it is quasicyclic by [19, Lemma 2.9]. Let L be the G-invariant nilpotent subgroup of G' such that G'/L is polycyclic-by-finite. Then $\mathrm{Aut}(G'/L)$ is polycyclic-by-finite by [21, Theorem 3.27]. Hence $C_G(G'/L) = G/L$, since G has no finite images, G/L is centre-by-locally cyclic so it is abelian. Therefore $G' = L$ is nilpotent. □

Theorem 3.11 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are* $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$*, then all proper subgroups of* G *are* $\mathfrak{N}(\mathfrak{P}\mathfrak{F})$*.*

Proof — Assume for a contradiction that the statement is false. In particular, G has no non-trivial finite images and an application of Theorem 3.7 and Lemma 3.10 yields that G is imperfect and G/G' is quasicyclic with G' is nilpotent. Suppose first that G' is periodic, then G is periodic and hence all it's proper subgroups of infinite rank are nilpotent-by-finite and the result follows by [12]. Thus G' is non-periodic and by [22, 5.2.6], $G'/\gamma_2(G')$ is non-periodic. Let $T/\gamma_2(G')$ be the torsion subgroup of $G'/\gamma_2(G')$. Hence G'/T is non-trivial, abelian and torsion-free. Applying [5, Lemma 2.3] we get that for each pair of primes p_1 and p_2, there exists a G-invariant subgroup M of G', such that $T < M$ and G'/M is an abelian $\{p_1, p_2\}$-group containing elements of orders p_1 and p_2. If the periodic group G/M has finite rank then all its proper subgroups are nilpotent-by-finite, and if G/M has infinite rank then all its proper subgroups are also nilpotent-by-finite by [12]. We deduce that G/M is either in $\mathfrak{NF}$ or it is a $MN\mathfrak{NF}$-group. Assume that G/M is in $\mathfrak{NF}$, then it is nilpotent. It follows, as G has no proper subgroup of finite index, that G/M is abelian, and so $G' = M$. Hence G/M is a $MN\mathfrak{NF}$-group. But the commutator subgroup of G/M cannot be a p-group for any prime p, contradicting [5, Corollary 2.7]. The statement is proved. □

Corollary 3.12 *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of infinite rank of G are in $\mathfrak{N}(\mathfrak{PF})$, then G is either in $\mathfrak{N}(\mathfrak{PF})$ or it is a $MN\mathfrak{NF}$-group.*

Proof — Application of Theorem 3.11 yields that all proper subgroups of G are in $\mathfrak{N}(\mathfrak{PF})$. Assume that G is not in $\mathfrak{N}(\mathfrak{PF})$. In particular, G has no proper subgroup of finite index and so it cannot be finitely generated. As finitely generated $\mathfrak{N}(\mathfrak{PF})$-groups are polycyclic-by-finite, G is locally (polycyclic-by-finite). Therefore G is a $MN\mathfrak{NF}$-group by [11, Theorem 2.13]. □

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