Groups whose Proper Subgroups of Infinite Rank are Minimax-by-Nilpotent or Nilpotent-by-Minimax *

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Abstract

Let $\mathcal{M}$ denote the class of of soluble-by-finite minimax groups, and $\mathcal{N}$ the class of nilpotent groups. The main result states that if $G$ is a group of infinite rank whose proper subgroups of infinite rank are $\mathcal{MN}$-groups, then $G$ is either in $\mathcal{MN}$ or it is a group of Heineken-Mohamed type, provided that $G$ satisfies a suitable generalized solubility condition. Moreover, we prove that a group of infinite rank all of its proper subgroups of infinite rank are $\mathcal{NM}$-groups is itself a $\mathcal{NM}$-group.

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1 Introduction

A group $G$ is said to have finite rank $r$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with such property. If there is no such $r$, the group $G$ has infinite rank. In recent years, generalized soluble groups $G$ of infinite rank whose proper subgroups of infinite rank belong to a given class $\mathcal{Y}$ have been studied and it was proved that all proper subgroups of $G$ are in the class $\mathcal{Y}$, sometimes these groups $G$

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themselves belong to \(Y\). The motivation of establishing such results arises from the classic theorems of Yu.I. Merzljakov [17] and V.P. Šunkov [23] concerning groups with rank restrictions on their abelian subgroups. We recall that a group \(G\) is locally graded if every non-trivial finitely generated subgroup of \(G\) has a non-trivial finite image. In [6], N.S. Černikov considered the class \(X\) obtained by taking the closure of the class of periodic locally graded groups by the closure operations \(P, \tilde{P}, R, L\). Clearly \(X\) is a subclass of the class of locally graded groups. In his paper [6], Černikov proved that an \(X\)-group of finite rank is (locally soluble)-by-finite. The \(X\)-groups form a large \(S\)-closed class of generalized soluble groups containing, in particular, the classes of locally (soluble-by-finite) groups, radical groups and residually finite groups. A result of B. Bruno and R.E. Phillips [4] shows that if \(G\) is an \(X\)-group of infinite rank in which all proper subgroups of infinite rank have finite commutator subgroup, then also the commutator subgroup \(G'\) of \(G\) is finite. Moreover, in [12], it is proved that a locally (soluble-by-finite) group \(G\) of infinite rank whose proper subgroups of infinite rank are abelian-by-finite, then all it’s proper subgroups are abelian-by-finite. The aim of the present paper is to provide a further contribution to the topic, replacing the terms “abelian group” by “nilpotent group” and “finite group” by “\(Y\)-group”, where \(Y\) is a subclass of the class of soluble-by-finite minimax groups with certain conditions of closure operations. As we will see, \(Y\) can be chosen to be the class \(C, P, F\), or \(M\) of Chernikov, polycyclic-by-finite or soluble-by-finite minimax groups, respectively.

Recall that a group \(G\) is minimal non-\(Y\), or briefly, \(MN\), if it is not a \(Y\)-group, but all its proper subgroups belong to \(Y\). The description of the structure of \(MN\)-groups can be considered as a first step in the investigation of groups of infinite rank whose proper subgroups of infinite rank belong to \(Y\).

Most of our notation is standard and can be found in [21].

2 The minimax-by-nilpotent case

In this section we are interested in the classes \(FN, CN, (PF)N, MN\). We shall give the complete characterization of \(X\)-groups of infinite rank in which all proper subgroups of infinite rank are in these classes. The knowledge of the structure of \(MN\)-groups (and
MNE\textsuperscript{N}-groups, MN(\textsuperscript{PF}\textsuperscript{N})-groups, MN\textsuperscript{MN}-groups) will be relevant in our considerations.

We record the following obvious result, which will be frequently used. Let \(\mathcal{Y}\) and \(\mathcal{Z}\) be two group classes, we say that \(\mathcal{Y}\) is \(\mathcal{Z}\)-characteristic (a generalization of the idea of [13]) if any \(\mathcal{Y}\mathcal{Z}\)-group \(G\) contains a characteristic \(\mathcal{Y}\)-subgroup \(A\) such that \(G/A\) is a \(\mathcal{Z}\)-group. Note that if \(\mathcal{Y}\) is any subgroup closed class, then \(\mathcal{Y}\) is \(\mathcal{N}\)-characteristic.

**Lemma 2.1** Let \(\mathcal{Y}\) and \(\mathcal{Z}\) be two group classes such that \(\mathcal{Y}\) is \(\mathcal{Z}\)-characteristic and \(\mathcal{N}_0\)-closed and \(\mathcal{Z}\) is \(\{H, \mathcal{N}_0\}\)-closed. Then \(\mathcal{Y}\mathcal{Z}\) is also \(\mathcal{N}_0\)-closed. Therefore if \(G\) is a MN\(\mathcal{Y}\mathcal{Z}\)-group, then every nilpotent image of \(G\) is a (possibly trivial) locally cyclic \(p\)-group for some prime \(p\).

**Proof** — Let \(H\) and \(K\) be normal \(\mathcal{Y}\mathcal{Z}\)-subgroups of a group \(G\). Then there exists two subgroups \(A\) and \(B\) such that \(A\) (resp. \(B\)) characteristic in \(H\) (resp. \(K\)), \(A, B\) are \(\mathcal{Y}\)-groups and \(H/A, K/B\) are \(\mathcal{Z}\)-groups. Clearly \(AB\) is a normal \(\mathcal{Y}\)-subgroup of \(G\) by the hypothesis. We also have that \(HK/AB\) is a \(\mathcal{Z}\)-group, as it is the product of \(HB/AB\) and \(KA/AB\). Hence \(HK\) is a \(\mathcal{Y}\mathcal{Z}\)-group. The rest of the claim follows by [19, Theorem 2.12].

Recall that a group \(G\) is of Heineken-Mohamed type [15], if \(G\) is infinite non-nilpotent with nilpotent and subnormal proper subgroups. The following result describe the structure of locally graded MN\(\mathcal{M}\)-groups. It shows in particular that such groups are precisely groups of Heineken-Mohamed type.

**Theorem 2.2** A group \(G\) is a locally graded MN\(\mathcal{M}\) if and only if it is a group of Heineken-Mohamed type.

**Proof** — Let \(G\) be a group as stated. Then, \(G\) cannot be finitely generated. As finitely generated \(\mathcal{M}\)-groups are obviously in \(\mathcal{M}\), it follows from the hypothesis that every finitely generated subgroup of \(G\) is an \(\mathcal{M}\)-group. So \(G\) is locally in \(\mathcal{M}\) and, by [9, Theorem], either \(G\) is a finite rank-by-nilpotent group or \(G/G'\) is \(p\)-quasicyclic for some prime \(p\) and every proper subgroup of \(G\) is nilpotent. Assume for a contradiction that \(G\) is finite rank-by-nilpotent. As every nilpotent image of \(G\) is locally cyclic \(p\)-group for some prime \(p\) by Lemma 2.1, it thus follows that \(G\) is of finite rank.

Let us suppose first that \(G\) contains a proper normal subgroup \(N\) of finite index. \(N\) belongs to \(\mathcal{M}\) and \(\gamma_c(N)\) is a \(\mathcal{M}\)-group for some positive integer \(c\); there is no loss of generality if we assume that \(N\)
is nilpotent. If $G$ is periodic then all its proper subgroups are Chernikov-by-nilpotent so, by [2, Corollary 2.1], $G$ belongs to $\mathcal{M}$. Therefore $G$ is non-periodic, so that $N$ is non-periodic. An application of [22, 5.2.6] shows that also the factor group $N/N'$ is non-periodic. Let $M/N'$ be a maximal free abelian subgroup of $N/N'$; then

$$(N/N')/(M/N')$$

is periodic. Since $G/N'$ has finite rank and $M/N'$ has a finite number of conjugates in $G/N'$, $(M/N')^G$ is a finitely generated abelian group. Furthermore, [2, Corollary 2.1] shows that the periodic group

$$(G/N')/(M/N')^G$$

is Chernikov-by-nilpotent hence, since every nilpotent image of $G$ is Chernikov, $(G/N')/(M/N')^G$ is a Chernikov group. It follows that $G/N'$ belongs to $\mathcal{M}$, and so $N/N'$ is an $\mathcal{M}$-group. Therefore $N$ belongs to $\mathcal{M}$, as $N$ is nilpotent. But this contradicts our assumption.

Thus $G$ has no proper subgroup of finite index. By [6], $G$ is locally soluble, and hence it has an ascending normal abelian series, so $G$ is hyperabelian. By [21, Theorem 9.31], the finite residual of $G$ is radicable nilpotent, and hence $G$ is nilpotent, since it has no proper subgroup of finite index. This contradiction gives the result. 

Now we turn to the study of groups of infinite rank in which all proper subgroups of infinite rank belong to $\mathcal{Y}$, where $\mathcal{Y}$ is an $S$-closed subclass of the class of soluble-by-finite minimax groups.

**Theorem 2.3** Let $\mathcal{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is $S$-closed. If $G$ is an $X$-group of infinite rank whose proper subgroups of infinite rank are $\mathcal{Y}$-groups, then all proper subgroups of $G$ are $\mathcal{Y}$-groups.

**Proof** — Clearly all proper subgroups of $G$ are finite rank-by-nilpotent. Using [9, Theorem], either $G$ is a finite rank-by-nilpotent group or $G/G'$ is $p$-quasicyclic for some prime $p$ and every proper subgroup of $G$ is nilpotent. In the latter case the result is immediate, so we may assume that $G$ is finite rank-by-nilpotent. Then $G$ contains a proper normal subgroup $N$ such that $G/N$ is nilpotent. Hence $G/G'$ is of infinite rank, and so $G$ contains a proper subgroup $M$ of infinite rank such that $G' \leq M$ and $G/M$ has infinite rank. Let $H$ be any subgroup of $G$ of finite rank. The product $HM$ is a proper subgroup of infi-
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finite rank of $G$, and hence belongs to $\mathcal{N}$, so $H$ also is a $\mathcal{N}$-group. Therefore all proper subgroups of $G$ are $\mathcal{N}$-groups, as desired. □

Next, we derive some results on $\mathcal{X}$-groups of infinite rank in which all proper subgroups of infinite rank are $\mathcal{F}$ (resp. $\mathcal{C}$, $(\mathcal{P}\mathcal{F})\mathcal{N}$, $\mathcal{M}$).

The class $\mathcal{F}$ of finite groups satisfies the hypotheses of Theorem 2.3; moreover, $\mathcal{M}\mathcal{N}$ of infinite rank are groups of Heineken-Mohamed type by [24, Theorem 3.5] and [4, Theorem 2]. The following result is valid.

**Corollary 2.4** Let $G$ be an $\mathcal{X}$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $\mathcal{F}$, then $G$ is either in $\mathcal{F}$ or it is a group of Heineken-Mohamed type.

Since the class $\mathcal{P}\mathcal{F}$ of polycyclic-by-finite groups satisfies the hypotheses of Theorem 2.3 so by [11, Theorem 3.9] and Corollary 2.4, we have the following consequence.

**Corollary 2.5** Let $G$ be an $\mathcal{X}$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $(\mathcal{P}\mathcal{F})\mathcal{N}$, then $G$ is either in $(\mathcal{P}\mathcal{F})\mathcal{N}$ or it is a group of Heineken-Mohamed type.

The class $\mathcal{C}$ of Chernikov groups satisfies the hypotheses of Theorem 2.3 which, together with [2] enables us to deduce the following consequence.

**Corollary 2.6** Let $G$ be $\mathcal{X}$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $\mathcal{C}$, then $G$ is either in $\mathcal{C}$ or it is a group of Heineken-Mohamed type.

The class $\mathcal{M}$ of soluble-by-finite minimax groups satisfies the hypotheses of Theorem 2.3, 2.2 so we have the following consequence.

**Corollary 2.7** Let $G$ be $\mathcal{X}$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $\mathcal{M}$, then $G$ is either in $\mathcal{M}$ or it is a group of Heineken-Mohamed type.

### 3 The nilpotent-by-minimax case

In this section, we consider the “dual” situation, that is the case in which proper subgroups of infinite rank of an $\mathcal{X}$-group of infinite rank
rank are $\mathcal{M}$ (resp. $\mathcal{M}(\mathcal{P})$, $\mathcal{NM}$). The results obtained should be seen in relation with the structure of $\mathcal{MN}\mathcal{C}$-groups (resp. $\mathcal{MN}\mathcal{M}(\mathcal{F})$-groups, $\mathcal{MN}\mathcal{MM}$-groups).

In [18, 3], it is shown that a locally graded group whose proper subgroups are in $\mathcal{M}$ is itself in $\mathcal{M}$. As a continuation of that work, we give the soluble-by-finite minimax $\mathcal{M}$ version by studying groups whose proper subgroups are $\mathcal{NM}$. The result will be accomplished by a series of lemmas.

**Lemma 3.1** Let $\mathcal{Y}$ be a class of groups contained in the class of finite rank groups which is $\mathcal{H}$-closed. If $G$ is a $\mathcal{NM}$-group, then $G$ has a characteristic nilpotent subgroup $N$ such that $G/N$ is a $\mathcal{Y}$-group.

**Proof** — Let $M$ be a normal nilpotent subgroup of $G$ such that $G/M$ is a $\mathcal{Y}$-group. Consider the characteristic closure of $M$ in $G$, and we write

$$
\overline{M} := \langle \sigma(M), \sigma \in \text{Aut}(G) \rangle.
$$

Then $\overline{M}$ is characteristic in $G$ and $G/\overline{M}$ is a $\mathcal{Y}$-group. Since $\sigma(M) \leq G$ and $\sigma(M) \cong M$ for all $\sigma \in \text{Aut}(G)$, $\overline{M}$ is generated by normal nilpotent subgroups and hence it locally nilpotent. Following the same way of the proof of [8, Lemma 1], we get that $\overline{M}$ is nilpotent. \[\square\]

**Lemma 3.2** Let $\mathcal{Y}$ be a class of groups contained in the class of finite rank groups and $\{\mathcal{H}, \mathcal{P}\}$-closed. If $G$ is a group containing a normal $\mathcal{NM}$-subgroup $N$ such that $G/N$ is in $\mathcal{Y}$, then $G$ is in $\mathcal{Y}$.

**Proof** — By Lemma 3.1, $N$ contains a characteristic nilpotent subgroup $L$ such that $N/L$ is a $\mathcal{Y}$-group. Since $G/N$ and $N/L$ are $\mathcal{Y}$-groups and the class $\mathcal{Y}$ is $\mathcal{P}$-closed, we deduce that $G/L$ belongs to $\mathcal{Y}$ and so $G$ is a $\mathcal{Y}$-group. \[\square\]

**Corollary 3.3** Let $G$ be a group whose proper normal subgroups belong to $\mathcal{NM}$. If $G$ is imperfect, then $G$ itself belongs to $\mathcal{NM}$.

**Proof** — If the quotient group $G/G'$ is decomposable, then $G = MN$ is a product of two proper normal subgroups $M$ and $N$. Since $M$ and $N$ are $\mathcal{NM}$-groups, $G$ is also a $\mathcal{NM}$-group, by Lemma 3.1 and Lemma 2.1. Now let $G/G'$ be an indecomposible group. Then $G/G'$ is Chernikov by [19, Lemma 2.9] and so $G$ lies in $\mathcal{NM}$ by Lemma 3.2. \[\square\]

**Lemma 3.4** Let $G$ be a locally nilpotent group whose proper subgroups are in $\mathcal{NM}$. Then $G \in \mathcal{NM}$. 


Proof — Assume for a contradiction that $G$ is a $\mathfrak{MN}$-group. It follows from Corollary 3.3 and Lemma 3.2 that $G$ is perfect and has no non-trivial finite images. Since the class of soluble groups is countably recognizable, $G$ is countable. Let $T$ be the torsion subgroup of $G$. If $G = T$ then, by [18, 3], $G$ lies in $\mathcal{EC}$, a contradiction. Hence $G \neq T$, and the factor group $G/T$ is countable locally nilpotent torsion-free. Since $G/T$ is the isolator of a proper subgroup $K/T$, $G/T$ itself is soluble by [14, Lemma 4.6]. This contradiction establishes the result. □

It is now easy to prove the following result.

**Theorem 3.5** Let $G$ be a locally graded group whose proper subgroups are in $\mathfrak{MN}$. Then $G \in \mathfrak{MN}$.

Proof — Assume for a contradiction that the statement is false. In particular, $G$ has no non-trivial finite images, see Lemma 3.2, and so $G$ is locally (soluble-by-finite). Moreover, we deduce from Corollary 3.3 and Lemma 3.4 that $G$ is perfect non-locally nilpotent. Let $V$ be the Hirsch-Plotkin radical of $G$. Clearly, the factor group $G/V$ is a $\mathfrak{MN}$-group. Replacing $G$ by $G/V$, it can be assumed without loss of generality that all ascendant subgroups of $G$ are of finite rank. It is easy to see, using [8, Theorem 2], that if $G$ has no simple images, then $G$ is radicable nilpotent. By this contradiction we have that $G$ contains a normal subgroup $M$ such that $\overline{G} = G/M$ is an infinite simple group. Since the class of (locally soluble)-by-finite groups is countably recognizable by [7, Lemma 3.5], and $\overline{G}$ has no non-trivial finite images, we deduce that $\overline{G}$ is countable. Let $\overline{K}$ be the super-inert subgroup of $\overline{G}$, as described in [10, Proposition 1]. Since $\overline{G}$ is not locally finite (see [18, 3]), the Hirsch-Plotkin radical of $\overline{K}$ is trivial (see [10, Theorem 2]). Hence $\overline{K}$ is finite and then, as $\overline{G}$ is periodic over $\overline{K}$, $\overline{G}$ is periodic. This implies the contradiction that $\overline{G}$ is locally finite. □

The last part of this section is devoted to the proof of results concerning groups whose proper subgroups of infinite rank $\mathcal{EC}$ (resp. $\mathcal{NP}$, $\mathfrak{MN}$).

**Lemma 3.6** If $G$ is a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are hypercentral-by-$\mathfrak{M}$, then $G$ is not simple.

Proof — Assume for a contradiction that $G$ is simple. Note that the proper subgroups of $G$ are either locally (soluble-by-finite) of finite rank or hypercentral-by-$\mathfrak{M}$. Since $G$ is locally (soluble-by-finite),
we deduce that the hypercentral-by-$\mathcal{M}$ subgroups are (locally soluble)-by-finite and, since the class (locally soluble)-by-finite is countably recognizable by [7, Lemma 3.5]. It follows that $G$ is countable, as $G$ has no finite images. Let $R$ be the super-inert subgroup of $G$, as described in [10, Proposition 1]. If $R$ has a non-trivial normal locally nilpotent subgroup then its Hirsch-Plotkin radical is non-trivial, and if $R$ has finite rank then its Hirsch-Plotkin radical is also non-trivial by [21, Lemma 10.39]. We deduce, using [10, Theorem 2], that $G$ is locally finite. By [16], $G$ is isomorphic to either $\text{PSL}(2, F)$ or $\text{Sz}(F)$ for some infinite locally finite field $F$. But each of these groups has a proper non hypercentral-by-Chernikov subgroup of infinite rank [20], a contradiction. Therefore the group $G$ is not simple.

**Theorem 3.7** Let $\mathcal{Y}$ be a subclass of the class of soluble-by-finite minimax groups which is $\{P, S, R_0\}$-closed. If $G$ is a perfect $\mathcal{X}$-group of infinite rank whose proper subgroups of infinite rank are in $\mathcal{N}_{\mathcal{Y}}$, then all proper subgroups of $G$ are in $\mathcal{N}_{\mathcal{Y}}$.

**Proof** — Assume for a contradiction that the statement is false. Let $N$ be a proper subgroup of finite index in $G$. So there is a normal nilpotent subgroup $L$ of $N$ such that $N/L$ is a $\mathcal{Y}$-group. Since $L$ has only finitely many conjugates in $G$ and $\mathcal{Y}$ is $\mathcal{R}_0$-closed, $N/L_G$ is a $\mathcal{Y}$-group, where, of course, $L_G$ is the core of $L$ in $G$. But then, as $L_G$ is nilpotent and $G/N$ belongs to $\mathcal{Y}$, $G$ is a $\mathcal{N}_{\mathcal{Y}}$-group. Hence $G$ has no non-trivial finite images and so it cannot be finitely generated. Now by [6], all proper subgroups of $G$ are (locally soluble)-by-finite; in particular $G$ is locally (soluble-by-finite).

Let $N$ be a proper normal subgroup of infinite rank of $G$. If $G/N$ has finite rank, then, in particular, all its proper subgroups are $\mathcal{M}$-groups, and hence $G/N$ itself a $\mathcal{M}$-group by Theorem 3.5. Then $G/N$ is soluble, a contradiction, because $G$ is perfect. Thus $G/N$ has infinite rank, so that if $H$ is any proper subgroup of finite rank of $G$, then $HN$ is a proper subgroup of infinite rank. It follows that $HN$ belongs to $\mathcal{N}_{\mathcal{Y}}$, and so $H$ is a $\mathcal{N}_{\mathcal{Y}}$-group. Therefore all proper normal subgroups of $G$ are of finite rank. Suppose that $G$ contains a normal subgroup $M$ such that $G = G/M$ is simple. But Lemma 3.6 shows that $G$ cannot be simple. Hence $G$ is of finite rank by [8, Theorem 2]. This contradiction completes the proof of the theorem. □

Now, by Theorem 3.7 and Theorem 3.5, we have the following corollary.
Corollary 3.8  Let $G$ be an $X$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $\mathcal{NM}$, then $G$ itself is in $\mathcal{NM}$.

Proof — Clearly we may suppose that $G$ is infinitely generated, see Lemma 3.2. So $G$ is locally (soluble-by-finite) and using [1, Lemma 3.1], we conclude that all proper normal subgroups are in $\mathcal{NM}$. By Corollary 3.3, one can assume that $G$ is perfect. Application of Theorem 3.7 yields that all proper subgroups of $G$ belong to $\mathcal{NM}$. So $G$ also is in $\mathcal{NM}$ by Theorem 3.5. \[\square\]

Following a similar approach using Theorem 3.7 and [18, 3], we deduce the following result.

Corollary 3.9  Let $G$ be an $X$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are in $\mathcal{NC}$, then $G$ itself is in $\mathcal{NC}$.

Now we consider non-perfect groups of infinite rank whose proper subgroups of infinite rank are in $\mathcal{NPF}$.

Lemma 3.10  If $G$ is a non-perfect group of infinite rank whose proper subgroups of infinite rank belong to $\mathcal{NPF}$. Then either $G \in \mathcal{NPF}$ or $G/G'$ is quasicyclic and $G'$ is nilpotent.

Proof — Suppose that $G$ does not belong to $\mathcal{NPF}$. In particular, $G$ has no non-trivial finite images and so it is locally (soluble-by-finite). By [1, Lemma 3.1], all proper normal subgroups of $G$ are in $\mathcal{NPF}$. If the factor group $G/G'$ is decomposable, then $G = MN$ is a product of two proper normal subgroups $M$ and $N$. It follows from Lemma 3.1 and Lemma 2.1 that $G$ is in $\mathcal{NM}$. Thus $G/G'$ is indecomposable, and hence it is quasicyclic by [19, Lemma 2.9]. Let $L$ be the $G$-invariant nilpotent subgroup of $G'$ such that $G'/L$ is polycyclic-by-finite. Then $Aut(G'/L)$ is polycyclic-by-finite by [21, Theorem 3.27]. Hence $C_G(G'/L) = G/L$, since $G$ has no finite images, $G/L$ is centre-by-locally cyclic so it is abelian. Therefore $G' = L$ is nilpotent. \[\square\]

Theorem 3.11  Let $G$ be an $X$-group of infinite rank. If all proper subgroups of infinite rank of $G$ are $\mathcal{NPF}$, then all proper subgroups of $G$ are $\mathcal{NPF}$.

Proof — Assume for a contradiction that the statement is false. In particular, $G$ has no non-trivial finite images and an application of Theorem 3.7 and Lemma 3.10 yields that $G$ is imperfect and $G/G'$ is quasicyclic with $G'$ is nilpotent. Suppose first that $G'$ is periodic, then $G$ is periodic and hence all it’s proper subgroups of infinite rank
are nilpotent-by-finite and the result follows by [12]. Thus \( G' \) is non-periodic and by [22, 5.2.6], \( G'/\gamma_2(G') \) is non-periodic. Let \( T/\gamma_2(G') \) be the torsion subgroup of \( G'/\gamma_2(G') \). Hence \( G'/T \) is non-trivial, abelian and torsion-free. Applying [5, Lemma 2.3] we get that for each pair of primes \( p_1 \) and \( p_2 \), there exists a \( G \)-invariant subgroup \( M \) of \( G' \), such that \( T < M \) and \( G'/M \) is an abelian \( \langle p_1, p_2 \rangle \)-group containing elements of orders \( p_1 \) and \( p_2 \). If the periodic group \( G/M \) has finite rank then all its proper subgroups are nilpotent-by-finite, and if \( G/M \) has infinite rank then all its proper subgroups are also nilpotent-by-finite by [12]. We deduce that \( G/M \) is either in \( \mathcal{N} \) or it is a \( \text{MN}\mathcal{N} \)-group. Assume that \( G/M \) is in \( \mathcal{N} \), then it is nilpotent. It follows, as \( G \) has no proper subgroup of finite index, that \( G/M \) is abelian, and so \( G' = M \). Hence \( G/M \) is a \( \text{MN}\mathcal{N} \)-group. But the commutator subgroup of \( G/M \) cannot be a \( p \)-group for any prime \( p \), contradicting [5, Corollary 2.7]. The statement is proved. \( \square \)

**Corollary 3.12** Let \( G \) be an \( \mathcal{X} \)-group of infinite rank. If all proper subgroups of infinite rank of \( G \) are in \( \mathcal{N}(\mathcal{P\mathcal{F}}) \), then \( G \) is either in \( \mathcal{N}(\mathcal{P\mathcal{F}}) \) or it is a \( \text{MN}\mathcal{N} \)-group.

**Proof** — Application of Theorem 3.11 yields that all proper subgroups of \( G \) are in \( \mathcal{N}(\mathcal{P\mathcal{F}}) \). Assume that \( G \) is not in \( \mathcal{N}(\mathcal{P\mathcal{F}}) \). In particular, \( G \) has no proper subgroup of finite index and so it cannot be finitely generated. As finitely generated \( \mathcal{N}(\mathcal{P\mathcal{F}}) \)-groups are polycyclic-by-finite, \( G \) is locally (polycyclic-by-finite). Therefore \( G \) is a \( \text{MN}\mathcal{N} \)-group by [11, Theorem 2.13]. \( \square \)

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