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# On Groups with Finite Hirsch Number

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**Abstract**

Suppose $G$ is a group with finite Hirsch number $h$ modulo the $k$-th term of its upper central series. The Hirsch number of the $k+1$-th term of the lower central series of $G$ is known to be finite and of order bounded in terms of $h$ and $k$. Here we give simpler proofs leading to simpler and sharper bounds. In particular and perhaps surprisingly, we show that the Hirsch number of the $h+2k+1$-th term of the lower central series of $G$ is bounded by $h(h+3)/2$; in particular it is bounded independently of $k$.

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## 1 Introduction

Let $G$ be any group. Then $G$ has finite Hirsch number the integer $h$ if $G$ has an ascending series exactly $h$ of whose factors are infinite cyclic, the remaining factors of the series being locally finite; $h$ here is independent of the choice of series. The upper central series of $G$ we denote by $\{\zeta_i(G)\}_{i\geqslant 0}$ and the lower central series of $G$ we denote by $\{\gamma^i G\}_{i\geqslant 1}$. We are interested here in the effect on $\gamma^{k+1}G$ of $G/\zeta_k(G)$ having finite Hirsch number, $k$ here and throughout the paper being an integer. Also the Hirsch number of $G$ we denote by $hn(G)$ and the (Prufer) rank (the upper bound over the finitely generated subgroups $X$ of $G$ of the minimal number of generators of $X$) by $rk(G)$.

Let $\eta_k(r)$ denote the least integer such that if G is a group with $hn(G/\zeta_k(G)) \leqslant r$ finite, then $hn(\gamma^{k+1}G) \leqslant \eta_k(r)$; we will see in a moment that $\eta_k(r)$ is an integer. If $k = 0$, then $G/\zeta_k(G) = G = \gamma^1 G$, so $\eta_0(r) = r$ for all $r \geqslant 0$. If $r = 0$, then $G/\zeta_k(G)$ is locally finite, so $\gamma^{k+1}G$ is locally finite (e.g. see [3] 4.21 and p.115) and hence $\eta_k(0) = 0$ for all $k \geqslant 0$. Also

$$r(r-1)/2 \leqslant \eta_1(r) \leqslant r(r+1)/2 \quad \text{for all } r \geqslant 0,$$

the first inequality coming from the free nilpotent group of class 2 on r generators and the second from Theorem 3.b) of [8]. Finally note that directly from its definition it follows that each $\eta_k(r)$ is monotonic increasing in r.

**Theorem 1** *Consider the integers* $k \geqslant 1$, $r \geqslant 0$ *and* $e = 2^{k-1}$. *Then*

$$\eta_k(r) \leqslant r.\eta_{k-1}(r) + \eta_{k-1}(r)^2 + \eta_1(r).$$

*Further if* $r \geqslant 4$, *then* $\eta_k(r) \leqslant 2^{e-1}\eta_1(r)^e \leqslant r^e(r+1)^e/2$ *and if* $r \leqslant 4$, *then* $\eta_k(r) \leqslant 2^{e-1}10^e$.

The first claim for $\eta_k(r)$ in Theorem 1 is the main part of the theorem. Clearly iteration of it for any specific k will produce a bound, but a very complicated bound, for $\eta_k(r)$ in terms of k, r and $\eta_1(r)$ that lies between $\eta_1(r)^e$ and, if $r \geqslant 4$, $2^{e-1}\eta_1(r)^e$. Theorem 7.1.25 of [1] also gives a bound $n_k(r)$ for $\eta_k(r)$ defined as follows: set

$$n_1(r) = r(5r^2 + 5r - 1)/2$$

and for each $k \geqslant 2$ set

$$n_k(r) = n_{k-1}(r)(5r^2(r+1)/2 + 5n_{k-1}(r)^2 + 5n_{k-1}(r) - 1).$$

It follows clearly from Theorem 1 that $\eta_k(r) \leqslant n_k(r)$ for all $k \geqslant 1$ and $r \geqslant 0$. Hence [1] Theorem 7.1.25 is an immediate corollary of our Theorem 1 above.

If G is a group with $hn(G/\zeta_k(G)) = r$ finite, then by Theorem 1 there exists N normal in G with G/N nilpotent of class bounded by k and hn(N) bounded, but in terms of both k and r. We can do things the other way round and choose N with the class of G/N bounded in terms of k and r and hn(N) bounded by a function of r only.

**Theorem 2** *Let* G *be a group and* k *and* r *non-negative integers with* $hn(G/\zeta_k(G)) \leqslant r$. *Then*

$$hn(\gamma^{2k+r+1}G) \leqslant r + \eta_1(r) \leqslant r(r+3)/2.$$

*Moreover unless* $k = r = 0$ *we have* $hn(\gamma^{2k+r}G) \leqslant r + \eta_1(r) \leqslant r(r+3)/2$.

Now 7.1.33 of [1] states that under the hypotheses of Theorem 2 the nilpotent residual $G^N$ of G has Hirsch number at most

$$r(5r^2 + 5r + 1)/2.$$

But $G^N$ trivially lies in $\gamma^{2k+r+1}G$, so [1] 7.1.33 is an immediate consequence of Theorem 2 and with the improved bound $r(r+3)/2$.

Below $\tau(G)$ denotes the unique maximal locally finite normal subgroup of a group G and $d(G)$ denotes the minimal number of generators of G, so

$$rk(G) = \sup\{d(X) : X \leqslant G, X \text{ finitely generated}\}.$$

Let G be a group with $\tau(G) = \langle 1 \rangle$. If $h = hn(G)$ is finite, then G is soluble-by-finite with $rk(G) \leqslant [7h/2] + 1$ (see Theorem 3.a) of [8]) and if G is soluble-by-finite with $r = rk(G)$ finite, then $hn(G) \leqslant 2r$. Thus we should expect some sort of analogue of Theorem 1 for rank.

Now $\langle \acute{\mathbf{P}}, \mathbf{L} \rangle(\mathfrak{A}\mathfrak{F})$ denotes the smallest class of groups containing all abelian groups and all finite groups that is closed under the ascending series operator $\acute{\mathbf{P}}$ and the local operator $\mathbf{L}$. It is in fact a very large class of groups indeed containing for example all locally soluble groups and all locally finite groups. Also it contains every group with finite Hirsch number. It is extensively studied in [8].

For non-negative integers k and r let $\mu_k(r)$ denote the least integer such that for all $\langle \acute{\mathbf{P}}, \mathbf{L} \rangle(\mathfrak{A}\mathfrak{F})$-groups G with $rk(G/\zeta_k(G)) \leqslant r$ we have $rk(\gamma^{k+1}G) \leqslant \mu_k(r)$. Here $\mu$ is the analogue of $\eta$. Clearly $\mu_0(r) = r$ for all $r \geqslant 0$ and $\mu_k(0) = 0 = \mu_k(1)$ for all $k \geqslant 1$. Also by Theorem 1.b) of [8] we have

$$\mu_1(r) \leqslant r(9r+1)/2 + m(m-1) \quad \text{for } m = r(1 - [-\log_2 r]) \leqslant r^2.$$

Now Kurdachenko and Otal in [2] prove that $\mu_k(r)$ is bounded by some recursively-defined function of k and r only. Here we present the following improvement (the bound in Theorem 3 comes from the recurrence relation $\mu_k(r) \leqslant r\mu_{k-1}(r) + \mu_1(r)$ while the bound in [2] effectively uses $\mu_k(r) \leqslant r\mu_{k-1}(r) + \mu_1(\mu_{k-1}(r)))$.

**Theorem 3** *For all* $k \geqslant 1$ *and* $r \geqslant 0$ *we have*

$$\mu_k(r) \leqslant (1 + r + r^2 + \ldots + r^{k-1})\mu_1(r).$$

**Theorem 4** *Let* $F$ *be any field not containing a square root of* $-1$ *and with* $\operatorname{char} F \neq 2$. *If* $G$ *is a 2-subgroup of* $GL(n, F)$, *then* $G$ *has rank at most* $n$.

Obviously the fields most of interest to us in this context are the rationals and the reals. Trivially for any field $F$ with $\operatorname{char} F \neq 2$, the general linear group $GL(n, F)$ contains a diagonal 2-subgroup of rank exactly $n$. The main theorem of [6] yields that if $P$ is a $p$-subgroup of $GL(n, F)$ with $p \neq \operatorname{char} F$, then $\operatorname{rk}(P) \leqslant n$ if $p$ is odd and $\operatorname{rk}(P) \leqslant [3n/2]$ if $p = 2$. Moreover if $F$ contains a square root of $-1$, then $GL(n, F)$ contains a 2-subgroup of rank exactly $[3n/2]$, cf. the example in [6]. Thus Theorem 4 nicely completes the picture for linear 2-subgroups. More to the point it enables us to make a little improvement to some of our bounds.

## 2 Proofs

Proof of Theorem 4 — We induct on $n$, the case $n = 1$ being trivial. Let $G$ be a finite 2-subgroup of $GL(n, F)$ where $n \geqslant 2$. It suffices to prove that $d = d(G)$ is at most $n$. Now $G$ is completely reducible by Maschke's theorem. If $G$ is not irreducible, then induction yields that $d \leqslant n$. Hence assume that $G$ is irreducible (over $F$).

Let $K$ be the algebraic closure of $F$. Then $G$ is monomial over $K$ ([5], 1.6 and 1.14), so there is a maximal diagonalizable (over $K$) normal subgroup $A$ of $G$ with $S = G/A$ isomorphic to a 2-subgroup of $\operatorname{Sym}(n)$. Now $V = F^{(n)}$ is an irreducible $FG$-module and hence is completely reducible as an $FA$-module. Let $U$ be an irreducible $FA$-submodule of $V$. Then $A/C_A(U)$ is cyclic (use Schur's lemma). Also

$$V = U \oplus Ug_2 \oplus \ldots \oplus Ug_m$$

for some $g_i$ in $G$. If $\dim_F U \geqslant 2$, then $d(A) \leqslant n/2$. Also $d(S) \leqslant n/2$ by the Proposition of [6] and so in this case $d \leqslant n$. Hence assume $\dim_F U = 1$. But then $A/C_A(U)$ embeds into $F^*$ and by hypothesis $F^*$

contains no element of order 4. Therefore $|A/C_A(U)| \leqslant 2$ and hence $A$ is an elementary abelian 2-group (of rank at most $n$).

Now $G$ is monomial over $K$, so $G$ embeds into the permutational wreath product $W = S(K^*)^{(n)}$ of $K^*$ by $S$. Clearly $K^* = E \times C$, where $C$ is a Prüfer $2^\infty$-group and $E$ is 2-free. It follows that the 2-group $G$ embeds into the permutation wreath product $SC^{(n)} \simeq W/E^{(n)}$ such that $A$ maps into $C^{(n)}$ and $G/A = S$ maps naturally onto $S \leqslant SC^{(n)}$. then $A$ maps into the $S$-submodule

$$B = \{b \in C^{(n)} : b^2 = 1\}$$

of $C^{(n)}$. Clearly $SB$ is the permutational wreath product of $\langle -1 \rangle$ by $S$ and hence $SB$ embeds into $\mathrm{Sym}(2n)$. Consequently $\mathrm{rk}(SB) \leqslant n$ by the Proposition of [6] again.

We have an embedding of $G$ into $SC^{(n)}$ that maps $A$ into $B$ and $G$ naturally onto $S$, but the image of $G$ in $SC^{(n)}$ may not lie in $SB$. However $G$ maps onto $G/A = S \leqslant SB$, so $A$ is an $S$-submodule of $B$. That is, $A$ is normalized by $S$ and $SA \leqslant SB$ has rank at most $n$. Thus

$$(SA : (SA)^2) \leqslant 2^n.$$

If $s \in S$ and $a \in A$, then $(sa)^2 = s^2[a, s]$ since $a^2 = 1$, so $(SA)^2 = S^2[A, S]$. But $[A, S] = [A, G] \leqslant G' \leqslant G^2$ and

$$\begin{aligned}(G : G^2) &\leqslant (G/A : (G/A)^2)(A : [A, G]) \leqslant (S : S^2)(A : [A, S]) \\ &\leqslant (SA : (SA)^2) \leqslant 2^n.\end{aligned}$$

Therefore $d \leqslant n$ and Theorem 4 follows. □

**Lemma 5** *Let* $G$ *be a locally nilpotent subgroup of* $GL(n, \mathbb{R})$ *with* $\mathrm{hn}(G) \leqslant h$. *Then* $\mathrm{rk}(G) \leqslant h + n$.

Proof — We may assume $G$ is finitely generated and hence nilpotent. Then $\tau(G)$ has finite rank at most $n$ by [6] and Theorem 4. Also $G/\tau(G)$ has all its upper central factors torsion-free. Therefore $G/\tau(G)$ has a series running from $\langle 1 \rangle$ to the whole group of length at most $h$ with all factors infinite cyclic. It follows that $\mathrm{rk}(G/\tau(G)) \leqslant h$ and $\mathrm{rk}(G) \leqslant h + n$. □

**Lemma 6** *Let* $G$ *be a group,* $A$ *an abelian normal subgroup of* $G$ *and* $Z \leqslant A$ *a central subgroup of* $G$. *Suppose* $\mathrm{hn}(G/C_G(A)) \leqslant r$ *and* $\mathrm{hn}(A/Z) \leqslant s$. *If also* $G/C_G(A/Z)$ *is locally nilpotent, then* $\mathrm{hn}([A, G]) \leqslant rs + s^2$.

Proof — Clearly we may assume $\tau(A) = \langle 1 \rangle$ and $Z = A \cap \zeta_1(G)$, so $A$ and $A/Z$ are both now torsion-free. Then $C_G(A/Z)/C_G(A)$ is torsion-free abelian and hence its rank is equal to its Hirsch number, $r_1$ say. Further $\mathrm{rk}(A/Z) = \mathrm{hn}(A/Z) \leqslant s$, so $G/C_G(A/Z)$ is isomorphic to a locally nilpotent linear group of degree $s$ over the rationals. Lemma 5 yields that

$$\mathrm{rk}(G/C_G(A/Z)) \leqslant r_2 + s,$$

where $r_1 + r_2 \leqslant r$. Therefore $\mathrm{rk}(G/C_G(A)) \leqslant r + s$. Finally Lemma 3.4.c) of [8] yields that

$$\mathrm{hn}([A, G]) = \mathrm{rk}([A, G]) \leqslant (r + s)s = rs + s^2.$$

The statement is proved. □

**Lemma 7** *For* $k \geqslant 1$ *let* $G$ *be a group with* $\mathrm{hn}(G/\zeta_k(G)) = r < \infty$ *and* $\mathrm{hn}((\gamma^k G)/(\gamma^k G \cap \zeta_1(G)) = s < \infty$. *Then*

$$\mathrm{hn}(\gamma^{k+1} G) \leqslant rs + s^2 + \eta_1(r).$$

*In particular, since the definition of* $\eta$ *applied to* $G/\zeta_1(G)$ *yields that* $s \leqslant \eta_{k-1}(r)$, *we have* $\mathrm{hn}(\gamma^{k+1} G) \leqslant r\eta_{k-1}(r) + \eta_{k-1}(r)^2 + \eta_1(r)$.

Proof — Always $[\zeta_k(G), \gamma^k G] = \langle 1 \rangle$, so $\gamma^k G \cap \zeta_k(G) \leqslant \zeta_1(\gamma^k G)$. Thus $\mathrm{hn}((\gamma^k G)/\zeta_1(\gamma^k G)) \leqslant r$ and $\mathrm{hn}((\gamma^k G)') \leqslant \eta_1(r)$. Set

$$A = (\gamma^k G)/(\gamma^k G)' \quad \text{and} \quad H = G/(\gamma^k G)'.$$

Then $\mathrm{hn}(H/C_H(A)) \leqslant r$, $H/C_H(A)$ is nilpotent (of class less than $k$) and $\mathrm{hn}(A/C_A(H)) \leqslant s$. Therefore $\mathrm{hn}([A, H]) \leqslant rs + s^2$ by Lemma 6. But $[A, H] = (\gamma^{k+1} G)/(\gamma^k G)'$. Consequently

$$\mathrm{hn}(\gamma^{k+1} G) \leqslant rs + s^2 + \eta_1(r)$$

and the statement is proved. □

Proof of Theorem 1 — Consider integers $k \geqslant$, $r \geqslant 0$, $e = 2^{k-1}$ and a group $G$ with $\mathrm{hn}(G/\zeta_k(G)) \leqslant r$. Suppose $\eta_{k-1}(r) < \infty$. By Lemma 7 there is an integer $s \geqslant 0$ with

$$s \leqslant \eta_{k-1}(r) \quad \text{and} \quad \mathrm{hn}(\gamma^{k+1} G) \leqslant rs + s^2 + \eta_1(r).$$

Therefore $\eta_k(r) \leqslant r.\eta_{k-1}(r) + \eta_{k-1}(r)^2 + \eta_1(r) < \infty$.

To prove our second bound for $\eta_k(G)$ we induct on k, this bound being trivial if $k = 1$. Suppose $k \geqslant 1$ and $r \geqslant 4$. Then

$$r + 1 < r(r-1)/2 \leqslant \eta_1(r)$$

and $e \geqslant 1$. Using induction on k and our first bound for $\eta_k(r)$ we have

$$\begin{aligned}\eta_{k+1}(r) &\leqslant r.2^{e-1}\eta_1(r)^e + 2^{2e-2}\eta_1(r)^{2e} + \eta_1(r)\\ &\leqslant (r+1)2^{e-1}\eta_1(r)^e + 2^{2e-2}\eta_1(r)^{2e}\\ &\leqslant 2^{e-1}\eta_1(r)^{e+1} + 2^{2e-2}\eta_1(r)^{2e} \leqslant 2.2^{2e-2}\eta_1(r)^{2e}.\end{aligned}$$

The second bound for $hn(\gamma^{k+1}G)$ with $r \geqslant 4$ follows.

For $r \leqslant 4$ we have $\eta_k(r) \leqslant \eta_k(4) \leqslant 2^{e-1}(4(4+1)/2)^e = 2^{e-1}.10^e$. □

**Lemma 8** *Let* F *be a perfect field,* G *a locally nilpotent group,* V *a (right)* FG*-module and* U *an* FG*-submodule of* V*. With* $\mathbf{g}$ *denoting the augmentation ideal of* G *in* FG*, assume* $U\mathbf{g}^k = \{0\}$ *for some integer* $k \geqslant 0$ *and suppose* $r = \dim_F V/U$ *is finite. Then there exists an* FG*-submodule* W *of* V *with* $\dim_F W \leqslant r$*, with* $V\mathbf{g}^{k+r} \leqslant W$ *and with* $U \cap W = \{0\}$.

Proof — If k or r is zero the claims are obvious, so assume otherwise. We may also assume that G acts faithfully on V. Suppose G is finitely generated. Set $\mathbf{a} = \mathrm{Ann}_{FG}(V/U)$. Then $V\mathbf{a}\mathbf{g}^k = \{0\}$ and

$$(\mathbf{a} \cap \mathbf{g})^{k+1} \leqslant \mathbf{a}\mathbf{g}^k \leqslant \mathbf{b} = \mathrm{Ann}_{FG}\, V.$$

Clearly $\dim_F(FG/(\mathbf{a} \cap \mathbf{g}))$ is finite (it's at most $r^2 + 1$). Hence the F-dimension of $FG/\mathbf{b}$ is finite (cf. §1 of [5] p.22 or [7] 4.7; it is here we use the assumption that G is finitely generated).

We may regard G as a subset of $R = FG/\mathbf{b}$. Let $G \leqslant G_u \times G_d$ be the Jordan decomposition of the (locally) nilpotent subgroup G in $\mathrm{End}_F R$ (see [5] 7.14, where $G_uG_d$ is denoted by $\mu(G)$, or 3.1.6 and 3.1.7 of [4]). Since F is perfect [4] 3.1.6 and its proof show that

$$G_uG_d \subseteq F[G] \leqslant R$$

(for each $g_d$ is constructed there as a polynomial in g, so $G_uG_d = GG_d$ is contained in F[G]); thus U is also an $FG_uG_d$-module. Now $G_u \subseteq R$ is unipotent and hence so is its image in $\mathrm{End}_F(V/U) \simeq F^{r\times r}$. Thus if $\mathbf{g}_u$

denotes the augmentation ideal of $G_u$ in $FG_u$, we have $V(\mathbf{g}_u)^r \leqslant U$; that is, $(\mathbf{g}_u)^r \leqslant \mathbf{a}/\mathbf{b}$.

If $g \in G$, then $g_d = (g_u)^{-1}g$ and $g_u$ and $g$ commute and act unipotently on $U$, the latter by hypothesis. Consequently $g_d$ acts unipotently on $U$ (see [5] 7.1.ii). Now $F[G_d] \leqslant R$ is semisimple Artinian by [5] 7.7 and 1.24.i).a), so $V$ is a direct sum of irreducible $FG_d$-modules. Let $U_1$ be the $G_d$-trivial homogeneous $FG_d$-component of $V$ and $W$ the sum of the remaining homogeneous $FG_d$-components of $V$. Since $[G_u, G_d] = \langle 1 \rangle$, so $U_1$ and $W$ are $R$-submodules of $V$. If $I$ is an irreducible $FG_d$-submodule of $U$, then $\dim_F I \leqslant \dim_F R < \infty$ and $G_d$ acts unipotently on $I$. If $\mathbf{g}_d$ denotes the augmentation ideal of $G_d$ in $FG_d$, then $I\mathbf{g}_d < I$, $I\mathbf{g}_d = \{0\}$ and $U\mathbf{g}_d = \{0\}$. Consequently $U \leqslant U_1$, $\dim_F W \leqslant \dim_F(V/U) = r$ and $U \cap W = \{0\}$. Further $G_d$ centralizes $V/W$ and $V(\mathbf{g}_u)^r \leqslant U$. Consequently $V\mathbf{g}^r \leqslant U \oplus W$ and therefore $V\mathbf{g}^{k+r} \leqslant W$.

We now drop the assumption that $G$ is finitely generated. Consider a finitely generated subgroup $H$ of $G$ with $\mathbf{h}$ denoting the augmentation ideal of $H$ in $FH$. By the above there exists an $FH$-submodule $W_H$ of $V$ with $V\mathbf{h}^{k+r} \leqslant W_H$ and $U \cap W_H = \{0\}$. Pick $W_H$ with $\dim_F W_H \leqslant r$ minimal. If $H \leqslant K \leqslant G$ with $K$ finitely generated and the obvious notations $\mathbf{k}$ and $W_K$, we have

$$\mathbf{h} \leqslant \mathbf{k} \quad \text{and} \quad V\mathbf{h}^{k+r} \leqslant W_H \cap W_K.$$

Thus by the choice of $W_H$ we have $W_H \leqslant W_K$.

Set $W = \bigcup_H W_H$. Then $W$ is an $FG$-submodule of $V$ and since each $\dim_F W_H \leqslant r$, so $\dim_F W \leqslant r$ and $W = W_H$ for some $H$. In particular $U \cap W = U \cap W_H = \{0\}$. Finally $\mathbf{g} = \bigcup_H \mathbf{h}$ and hence

$$V\mathbf{g}^{k+r} \leqslant \bigcup_H V\mathbf{h}^{k+r} \leqslant \bigcup_H W_H = W.$$

The proof is complete. □

Proof of Theorem 2 — Thus let $G$ be a group and $k$ and $r$ non-negative integers such that $\mathrm{hn}(G/\zeta_k(G)) \leqslant r$. If $k = 0$ then

$$\mathrm{hn}(\gamma^1 G) = \mathrm{hn}(G) \leqslant r$$

and the claims follow in this case, whether or not $r$ is 0. Suppose $k \geqslant 1$. If $r = 0$ then $G/\zeta_k(G)$ is locally finite and hence $\gamma^{k+1}G$ is also locally finite. Thus $\text{hn}(\gamma^{k+1}G) = 0$ and $2k + r \geqslant k + 1$. Again the claims follow. Now assume both $k$ and $r$ are positive. Always $[\zeta_k(G), \gamma^k G] = \langle 1 \rangle$, so $\text{hn}((\gamma^k G)/\zeta_1(\gamma^k G)) \leqslant r$. Therefore

$$\text{hn}((\gamma^k G)') \leqslant \eta_1(r).$$

Let $T/(\gamma^k G)' = \tau((\gamma^k G)/(\gamma^k G)')$, $A = (\gamma^k G)/T$ and $H = G/T$. If $B = (\gamma^k G \cap \zeta_k(G))T/T$, then $[B, {}_k H] = \langle 1 \rangle$. Note that $H/C_H(A)$ is nilpotent.

Now $A$ is torsion-free abelian. Let $V = \mathbb{Q}A$ and $U = \mathbb{Q}B$, where $\mathbb{Q}$ is the field of rational numbers and we are taking tensor products over $\mathbb{Z}$. Since $\text{hn}(A/B) \leqslant r$, so $\dim_{\mathbb{Q}}(V/U) \leqslant r$. Also if $\mathbf{g}$ denotes the augmentation ideal of $G$ in $\mathbb{Q}G$, then $U\mathbf{g}^k = \{0\}$. By Lemma 8 there is a $\mathbb{Q}G$-submodule $W$ of $V$ with $\dim_{\mathbb{Q}} W \leqslant r$ and $V\mathbf{g}^{k+r} \leqslant W$. Set $C = A \cap W$. Then $\text{hn}(C) = \text{rk}(C) \leqslant r$ and $[A, {}_{k+r}G] \leqslant C$. Define $D \leqslant G$ by $D/T = C$. Then

$$\text{hd}(D) \leqslant r + \eta_1(r) \quad \text{and} \quad \gamma^{2k+r}G \leqslant [\gamma^k G, {}_{k+r}G] \leqslant D.$$

The proof is complete. □

**Lemma 9** *For integers* $k \geqslant 2$ *and* $r \geqslant 1$, *let* $G$ *be a* $\langle \acute{\boldsymbol{P}}, \boldsymbol{L} \rangle(\mathfrak{A}\mathfrak{F})$*-group with* $\text{rk}(G/\zeta_k(G)) \leqslant r$ *and* $\text{rk}((\gamma^k G)/\gamma^k G \cap \zeta_1(G)) = s$. *Then*

$$\text{rk}(\gamma^{k+1}G) \leqslant rs + \mu_1(r).$$

*In particular,* $\text{rk}(\gamma^{k+1}G) \leqslant r\mu_{k-1}(r) + \mu_1(r)$ *since by definition of* $\mu$ *we have* $s \leqslant \mu_{k-1}(r)$.

Proof — Always $[\zeta_k(G), \gamma^k G] = \langle 1 \rangle$ and $\gamma^k G \cap \zeta_k(G) \leqslant \zeta_1(\gamma^k G)$. Hence $\text{rk}((\gamma^k G)/\zeta_1(\gamma^k G)) \leqslant r$ and $\text{rk}((\gamma^k G)') \leqslant \mu_1(r)$. Set

$$A = (\gamma^k G)/(\gamma^k G)' \quad \text{and} \quad H = G/(\gamma^k G)'.$$

Then $\text{rk}(H/C_H(A)) \leqslant r$ and $\text{rk}(A/C_A(H)) \leqslant s$. By Lemma 3.4.c) of [8] we have $\text{rk}([A, H]) \leqslant rs$. But $[A, H] = (\gamma^{k+1}G)/(\gamma^k G)'$. Therefore $\text{rk}(\gamma^{k+1} \leqslant rs + \mu_1(r)$. □

Proof of Theorem 3 — If $r = 0$ the claim is $0 \leqslant 0$ and if $k = 1$ the claim is $\mu_1(r) \leqslant \mu_1(r)$. Thus assume $r \geqslant 1$ and $k \geqslant 2$. By Lemma 9 we have $\mu_k(r) \leqslant r.\mu_{k-1}(r) + \mu_1(r)$. Then induction on $k$ yields that

$$\begin{aligned}\mu_k(r) &\leqslant r(1 + r + \ldots + r^{k-2})\mu_1(r) + \mu_1(r)\\ &= (1 + r + r^2 + \ldots + r^{k-1})\mu_1(r)\end{aligned}$$

and the theorem is proved. □

The following is an easy consequence of Theorem 2.

**Corollary 10** *If* G *is a* $\langle \acute{\boldsymbol{P}}, \boldsymbol{L}\rangle(\mathfrak{A}\mathfrak{F})$*-group with*

$$\tau(G) = \langle 1 \rangle \quad \text{and} \quad \mathrm{rk}(G/\zeta_k(G)) \leqslant r$$

*then* $\mathrm{rk}(\gamma^{2k+r+1}G) \leqslant 7r^2 + 10r + [r/2] + 1$.

Proof — By Theorem 1.a) of [8] (and [3] 4.21 Corollary 2) G is soluble-by-finite. Thus $\mathrm{rk}(G/\zeta_k(G)) \leqslant r$ implies $\mathrm{hn}(G/\zeta_k(G)) \leqslant 2r$. Hence

$$\mathrm{hn}(\gamma^{2k+r+1}G) \leqslant r(2r + 3)$$

by Theorem 2 and therefore $\mathrm{rk}(\gamma^{2k+r+1}G) \leqslant 7r^2 + 10r + [r/2] + 1$ by [8] Theorem 3.a). □

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