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Almost Fixed-Point-Free Automorphisms of Prime Power Order

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To Hermann on his Eightieth Birthday

Abstract

We study the effect under various rank restrictions of a group having an automorphism of prime power order whose fixed-point set is also finite of prime power order for the same prime. Generally our conclusions are that the group has a soluble normal subgroup of bounded derived length. Not surprisingly the bound gets larger as the rank restrictions get weaker.

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1 Introduction

We study the effect on a group satisfying rank, sometimes very weak rank, restrictions of having an automorphism of finite order and with few fixed points. The central case in our discussion is that of a finite extension of a soluble FAR group. These are defined and discussed in [7], effectively as groups with series of finite length each factor of which is either finite or abelian with finite torsion-free rank and finite q-rank for every prime q. By a theorem, in fact by a much more general theorem of Baer and Heineken [1], these are exactly the soluble-by-finite groups whose abelian subgroups have finite torsion-free rank and finite q-rank for every prime q.

It is more convenient here, both for the proofs and since it suggests fruitful generalizations, to work with a different but equivalent definition. A soluble-by-finite group G is FAR if (and only if) it has finite Hirsch number and satisfies min-q for every prime q. A group G has Hirsch number h if G has a series of finite length with exactly h of the factors infinite cyclic, the remaining factors of the series being locally finite; G satisfies min-q if it satisfies the minimal condition on q-subgroups.

Let G be a group with finite Hirsch number h and ϕ an automorphism of G of finite order m and with its fixed-point $C_G(\phi)$ of finite order c. If $m = p$ a prime then by Theorem D of [2] the group G has a ϕ-invariant nilpotent normal subgroup of finite index (in fact index bounded in terms of h, p and c only) of nilpotency class at most $k(p)$, where $k(m) \leqslant 2^{m-1}$ denotes the Kreknin integer-valued function of m only, see [6], especially pages 83 and 94). Further if $m = 2^2$ then in particular G is a finite extension of a soluble group of derived length at most 3 whenever G is (torsion-free)-by-finite, see [8]. Thus our main focus here will be on the case where m is not a prime but is a power of a prime.

Our first theorem gives a general method for lifting results on automorphisms of finite groups to results on automorphisms of certain groups with finite Hirsch number. Unfortunately our, or rather my, knowledge of finite groups with automorphism with few fixed points is sufficiently limited that I have found Theorem 1 below less useful that I would have hoped. It is, however, the basis of our proof of Theorem 2 below.

Theorem 1.1 *Let G be a group with finite Hirsch number h satisfying min-q for every prime q. Let ϕ be an automorphism of G with $m = |\phi|$ and $c = |C_G(\phi)|$ both finite. Let H be any subgroup of G of finite index. Then there exist characteristic subgroups L and M of G with $L \leqslant M \leqslant H$ such that G/L is finite, $M^n \leqslant L$ for $n = m^h$ and $C_{G/L}(\phi) \leqslant C_G(\phi)M/L$.*

Thus for example if m and c are π-numbers for some set π of primes then $C_{G/L}(\phi)$ is also a π-group. Notice that G in Theorem 1 need not be soluble-by-finite. If for finite groups G and suitable m and c we knew that G must lie in a variety $\mathbf{V}$, then for residually finite (or even just locally residually finite) G as in Theorem 1 we would hope to deduce that G also lies in $\mathbf{V}$. For $\mathbf{V}$ we have in mind, for example, the variety of soluble groups of derived length at most d. The following is the main result of this paper. For any group G, we denote its unique maximal, locally finite, normal subgroup by $\tau(G)$.

Theorem 1.2 *Let G be a group with finite Hirsch number satisfying min-q for every prime q. Let ϕ be un automorphism of G and $m = p^{\mu}$ and $c = p^{\gamma}$ powers of the prime p suche that $\phi^m = 1$ and $|C_G(\phi)| = c$. Set $\mu' = \mu$ if p is odd; if $p = 2$ set $\mu' = 2\mu - 2$ if $m > 2$ and $\mu' = 1$ otherwise. The following hold.*

a) *If $\tau(G)$ is finite and if G is a finite extension of a residually finite p-group, then G is a finite extension of a soluble group with derived length at most $2k(m)$, where $k(m)$ is the Kreknin function.*

b) *If $\tau(G)$ is finite, then G is a finite extension of a soluble group of derived length at most $2k(m) + 1$.*

c) *If $\tau(G)$ is a Chernikov group, then G is a finite extension of a soluble group of derived length at most $2k(m) + 2$.*

d) *If $O_{p'}(G)$ is finite, then G is a finite extension of a soluble group of derived length at most $2k(m) + 2$.*

e) *If $G = O_{p'}(G)$, then G has a characteristic subgroup S of finite index such that S has a characteristic series of length μ' with all its factors hypercentral groups. If G is locally soluble we may choose $S = G$.*

f) *The group G has a characteristic series with its first μ' factors hypercentral p' groups, its next $2k(m) + 2$ factors abelian and its top factor finite.*

Notice that this theorem covers all finite extensions of soluble FAR groups. Further Part b) covers all polycyclic groups and Part c) covers all soluble-by-finite FATR groups and hence also all soluble-by-finite minimax groups, see [7] Page 86 for definitions. In connection with Part a), various equivalent conditions for G to be a finite extension of a residually finite-p group are given by the theorem of [10]. The main claim of Part f) can be expressed more succinctly in P. Hall's symbolic notation as $G \in (L\mathbf{N})^{\mu'}\mathbf{A}^{2k(m)+2}\mathbf{F}$. The factors in the series in f) all have central height at most $\omega 2$. At the expense of extending the series by an additional abelian factor we can arrange for all these factors to have central height at most ω.

In Theorem 1.2 suppose for a particular $m = p^{\mu}$ there exists a positive integer $d(m)$ such that for every prime $q \neq p$ if a finite $q-$subgroup Q has a fixed-point-free automorphism of order m, then Q has derived length at most $d(m)$ (and of course such a $d(m)$ does exist if $m = p$ or if $m = 4$). Then one can easily deduce from Theorem 1.2 that G in Theorem 1.2 has a soluble characteristic subgroup of finite index and derived length at most $d(m)\mu' + 2k(m) + 2$. At the end of this paper we discuss the special case where $p = 2$ and $m = 4$.

2 The Proofs

In the main we work with weaker hypotheses than Theorems 1.1 and 1.2 might suggest. In particular Lemmas 2.5 and 2.7 below are effectively Theorems 1.1 and 1.2 but with weaker hypotheses.

Lemma 2.1 *Let ϕ be an automorphism of a group G and S $\phi-$invariant, locally finite, normal subgroup of G. If $C_S(\phi) = \langle 1 \rangle$ and if $\phi|_S$ has finite order, then $C_{G|S}(\phi) = C_G(\phi)S/S$.*

Proof — See [9] Proposition 14b). □

Lemma 2.2 *Let A a periodic divisible abelian normal subgroup of a group G with A having finite $q-$rank for every prime q. Suppose ϕ is an automorphism of G with $A\phi = A$ and with $C_A(\phi)$ finite. Then $C_{G/A}(\phi) = C_G(\phi)A/A$.*

Proof — Let $\gamma : g \mapsto g^{-1}(g\phi)$ for $g \in G$, $C = C_G(\phi)$ and $K/A = C_{G/A}(\phi)$. Now $\gamma|_A$ is an endomorphism of A, so $A\gamma$ is divisible and $A = A\gamma \times B$ for some divisible subgroup B of A. Further $C \cap A$ is finite and $A\gamma \simeq A/(C \cap A)$. But A and $A\gamma$ are both periodic divisible abelian and now with the same finite $q-$rank for every prime q. Consequently $B = \langle 1 \rangle$ and $A\gamma = A$.

Let $k \in K$. Then $k\gamma \in A$, so $k\gamma = a\gamma$ for some a in A. Thus

$$k^{-1}(k\phi) = a^{-1}(a\phi), (ka^{-1})\phi = ka^{-1} \text{ and } ka^{-1} \in C.$$

Therefore $K \leqslant CA$. Trivially $CA \leqslant K$. □

Lemma 2.3 *Let ϕ be an automorphism of a group G and m a positive integer with $\phi^m = 1$. Suppose A is an abelian ϕ-invariant normal subgroup of G and set $C = C_G(\phi)$ and $K = \{g \in G : g^{-1}(g\phi) \in A^m\}$. If $A \cap C$ is m-torsion free, then $K = C(A \cap K)$.*

Proof — See [9] Lemma 15. □

Lemma 2.4 *Let* $\langle 1\rangle \leqslant T = H_0 < H_1 < \cdots < H_r = H \leqslant G$ *be a characteristic series of the group* G *such that* T *is a finite group of order dividing* t *and exponent dividing* e *and each* H_i/H_{i-1} *is torsion-free abelian of finite rank. Let* h *bound above the Hirsch number* $\sum_i rank(H_i/H_{i-1})$ *of* H. *Suppose* m *is a positive integer and* ϕ *is an automorphism of* G *with* $\phi^m = 1$. *Then there exist integer-valued functions* $s(h, m, t)$ *and* $n(r, m, e)$ *of the exhibited variables only and characteristic subgroups* $L \leqslant M$ *of* H *with*

$$(H : M) \leqslant s(h, m, t), \quad M^{n(r,m,e)} \leqslant L \quad \textit{and} \quad C_{G/L}(\phi) \leqslant C_G(\phi)M/L.$$

Note that $r \leqslant h$ and e divides t, so if we wish we may replace $n(r, m, e)$ by a function $n(h, m, t)$ with the same variables as s.

Of course Lemma 2.4 is only of interest if $H < G$. There is in general no best choice for s and n in that there is scope to increase/decrease s by decreasing/increasing n. As a trivial example suppose $r = 0$. Then we may choose $M = H$, $L = \langle 1\rangle$, $s(0, m, t) = 1$ and $n(0, m, e) = e$. Alternatively we may choose $M = L = \langle 1\rangle$, $s(0, m, t) = t$ and $n(0, m, e) = 1$. The proof of Lemma 2.4 can be used to arrive at the small value $n(r, m, e) = m^r$, which is independent of e note, but this requires a really large choice for $s(h, m, t)$. Our proof below uses induction on r.

Proof — Using induction on r we construct $n(r, m, e)$ and also a function $s(r; h, m, t)$ bounding $(H : M)$. Trivially $0 \leqslant r \leqslant h$, so we then set

$$s(h, m, t) = \max_{0\leqslant r\leqslant h} s(r; h, m, t).$$

If $r = 0$ set $M = L = \langle 1\rangle$, $s(0; h, m, t) = t$ and $n(0, m, e) = 1$. Suppose $r > 0$ and assume we have defined $s(r-1; h, m, t)$ and $n(r-1, m, e)$ for all relevant values of h, m, t and e. Set $C = C_G(\phi)$.

If K denotes $C_G(T)$, then clearly $(G : K)$ divides $t!$ and $H_1 \cap K$ is nilpotent of class at most 2. Also $H_1 \cap K$ is (central of exponent dividing e)-by-(torsion-free abelian). Consequently if $f = e^2$, then $X = (H_1 \cap K)^f$ is torsion-free abelian. Then with $Y = (H_1 \cap K)^{fm} = X^m$, Lemma 2.3 yields that

$$C_1 = C_{G/Y}(\phi) \leqslant CX/Y \leqslant C(H_1 \cap K)/Y.$$

In particular if $r = 1$ we may set $L = Y$ and $M = X$, with $s(1; h, m, t) = (t!)t^{2h+1}$ and $n(1, m, e) = m$. Alternative we may choose $M = H_1 \cap K$, $s(1; h, m, t) = t!$ and $n(1, m, e) = e^2m$.

Now assume that $r \geqslant 2$. Clearly $(H_1 \cap K)/Y$ has exponent dividing fm and order dividing $t(fm)^h$. Apply induction on r to $(H\cap K)/Y \leqslant G/Y$. Then there exist characteristic subgroups $L \leqslant M_1$ of $H\cap K$ with $Y \leqslant L$,

$$(H\cap K : M_1) \leqslant s(r-1; h, m, t(fm)^h),$$
$$(M_1)^{n(r-1,m,fm)} \leqslant L,$$
$$C_{G/L}(\phi) \leqslant C_1(M_1/Y)/(L/Y) \leqslant CXM_1/L.$$

Thus set $M = XM_1$, $s(r; h, m, t) = s(r-1; h, m, t(fm)^h)(t!)$ and $n(r, m, e) = n(r-1, m, e^2m)m$ (alternatively set $M = (H_1 \cap K)M_1$ and $n(r, m, e) = n(r-1, m, e^2m)e^2m$). In particular, with the first choice of M, if $n(r-1, m, e) = m^{r-1}$, then $n(r, m, e) = m^r$. □

Let G be a group with finite Hirsch number at most h and with $\tau(G)$ (locally soluble)-by-finite. Suppose G satisfies min-p for all p in some finite set π of primes. Then G has a characteristic series

$$\langle 1\rangle \leqslant S \leqslant T = H_0 < H_1 < \cdots < H_r = H \leqslant G,$$

where S is a π'-group ($= O_{\pi'}(H)$ in fact), T/S is a divisible abelian π-group of finite rank, each H_i/H_{i-1} is torsion-free abelian (necessarily of finite rank; also $T = \tau(H)$) and G/H is finite (use [5] 3.17 & 3.13 and [9] Lemmas 4 & 6).

Lemma 2.5 *With the notation above let ϕ be an automorphism of G and m and c (finite) π-numbers with $\phi^m = 1$ and $|C_G(\phi)| = c$. Then G has characteristic subgroups L and M with $T \leqslant L \leqslant M \leqslant H$ such that $(G : L)$ is finite, $M^n \leqslant L$ for $n = m^r$ and $C_{G/L}(\phi) \leqslant C_G(\phi)M/L$. Further $C_{G/L}(\phi)$ is a finite π-group; indeed its order divides cm^{rh}.*

Proof — Clearly $C_S(\phi) = \langle 1\rangle$. Hence by Lemma 2.1 we have $C_{G/S} = C_G(\phi)S/S$. Then by Lemma 2.2 we have $C_{G/T}(\phi) = C_{G/S}(T/S)/(T/S) = C_G(\phi)T/T$. Consequently by Lemma 2.4 and the comments there after there exist characteristic subgroups L and M of G with $T \leqslant L \leqslant M \leqslant H$, $(G : L)$ finite, $M^n \leqslant L$ and $C_{G/L}(\phi) \leqslant C_{G/T}(\phi)(M/T)/(L/T) = C_G(\phi)M/L$. Clearly the order of $C_{G/L}(\phi)$ divides $|C_G(\phi)|(M : L)$. The latter divides the π-number cm^{rh}. The proof of the lemma is complete. □

Lemma 2.6 *Let G be a group with finite Hirsch number. Suppose ϕ is an automorphism of G of finite order m and m is a power of the prime p. If $C_G(\phi)$ satisfies min-p, then G also satisfies min-p.*

Proof — Suppose that $m \geqslant p$ and that $C_G(\phi^{m/p})$ satisfies min-p (which we know holds true if $m = p$). Then $\phi^{m/p}$ has order p and hence G satisfies min-p by [5] 3.2. Clearly ϕ has order dividing m/p on $C_G(\phi^{m/p}) \geqslant C_G(\phi)$. A simple induction on m completes the proof. □

Lemma 2.7 *Let G be a group with finite Hirsch number at most h. Suppose p is a prime, ϕ an automorphism of G and $m = p^\mu$ and $c = p^\gamma$ powers of p such that $\phi^m = 1$ and $|C_G(\phi)| = c$. Set $\mu' = \mu$ if p is odd; if $p = 2$ set $\mu' = 2\mu - 2$ if $m > 2$ and $\mu' = 1$ otherwise. Assume $\tau(G)$ is (locally soluble)-by-finite. Then the following hold.*

- *a) If $\tau(G)$ is finite and if G is a finite extension of a residually finite p-group, then G is a finite extension of a soluble group with derived length at most $2k(m)$, where $k(m)$ is the Kreknin function.*
- *b) If $\tau(G)$ is finite, then G is a finite extension of a soluble group of derived length at most $2k(m) + 1$.*
- *c) If $\tau(G)$ is a Chernikov group, then G is a finite extension of a soluble group of derived length at most $2k(m) + 2$*
- *d) If $O_{p'}(G)$ is finite, then G is a finite extension of a soluble group of derived length at most $2k(m) + 2$.*
- *e) If $G = O_{p'}(G)$, then G has a characteristic subgroup S of finite index such that S has a characteristic series of length μ' with all its factors locally nilpotent. If G is locally soluble we may choose $S = G$.*
- *f) The group G has a characteristic series with its first μ' factors locally nilpotent p'-groups, its next $2k(m) + 2$ factors abelian and its top factor finite.*

Proof — By Lemma 6 the group G satisfies min-p.

a) Here G has a characteristic subgroup N of finite index that is residually finite-p and has a characteristic series

$$\langle 1 \rangle = N_0 < N_1 < \cdots < N_r = N$$

with each factor N_i/N_{i-1} torsion-free and abelian (we are using here that $\tau(G)$ is finite and also Lemmas 4 & 6 of [9]). We prove that there is a positive integer e, in fact depending only on p, μ, γ, & h, such that N^e is soluble with derived length at most $2k(m)$. Since $(G : N^e)$ is clearly finite (its order divides $(G : N)e^h$), the proof of Part a) will be complete.

Let H be any characteristic subgroup of N with N/H a finite p-group. By Lemma 5 there exist characteristic subgroups $L \leqslant M$ of H with $(G : L)$ finite, with $M^n \leqslant L$ for $n = m^r$ and with

$$C_{G/L}(\phi) \leqslant C_G(\phi)M/L.$$

Then the order of $C_{G/L}(\phi)$ divides p to the power of $\gamma + \mu r h$. There exists a Sylow p-subgroup P/L of G/L normalized by ϕ (since the number of such Sylow subgroups is congruent to 1 modulo p). Hence P contains a normal subgroup $Q \geqslant L$ such that $(P : Q)$ is (p, μ, γ, h)-bounded and such that Q/L has derived length at most $2k(m)$ by 12.15 of [6]. Finally P covers N/H, so there exists an integer e independent of the choice of H such that $N^e H/H$ has derived length at most $2k(m)$. But $\cap_H H = \langle 1 \rangle$. Therefore N^e is soluble of derived length at most $2k(m)$, thus completing the proof of Part a).

b) Denote the d-th derived subgroup of a group D by $D^{(d)}$. There exist subgroups $R \leqslant N$ of G with R torsion-free nilpotent, N/R finitely generated, abelian and G/N finite (since $\tau(G)$ is finite, by Lemmas 4 & 6 of [9] G is soluble-by-finite and thus we can apply [7] 5.2.2 and 5.2.3). If X is a finite subset of R, then $Y = \langle X\langle\phi\rangle\rangle$ is finitely generated, torsion-free and nilpotent. Therefore Y is residually a finite p-group. Thus we can apply the proof of part a) to Y, choosing Y itself for the subgroup N there. Hence there exists a positive integer e, dependent on G but not on the choice of X, such that $(Y^e)^{(2k(m))} = \langle 1 \rangle$. This is for all such X and hence $(R^e)^{(2k(m))} = \langle 1 \rangle$. But R/R^e is finite, for example of order dividing e^h. Hence G/R^e is finitely generated and abelian-by-finite. Therefore there is a characteristic subgroup $S \geqslant R^e$ of G with S/R^e abelian and G/S finite. Clearly S is soluble of derived length at most $2k(m) + 1$. Part b) is proved.

c) Let D denote the minimal subgroup of $\tau(G)$ of finite index. Then D is abelian and $\tau(G/D)$ is finite. By Part b) and Lemma 2 the group G/D is a finite extension of a soluble group with derived length at most $2k(m) + 1$. Therefore G is a finite extension of a soluble group of derived length at most $2k(m) + 2$.

d) Now $\tau(G)$ is (locally soluble)-by-finite, so $(\tau(G) : O_{p'p}(G))$ is finite by min-p and [5] 3.17 and $O_{p'p}(G)/O_{p'}(G)$ is Chernikov. Therefore $\tau(G)$ is Chernikov and Part d) follows from Part c).

e) Clearly here $C_G(\phi) = \langle 1 \rangle$. By hypothesis G has a locally soluble normal subgroup S of finite index and we may choose such S with $S = S\phi$ (if G is locally soluble choose $S = G$). Let **X** denote the set of ϕ—invariant finite subgroups of S.
Since G here is locally finite and $|\phi|$ is finite, **X** is a local system of S. For $X \in \mathbf{X}$ let S_X denote the set of all normal series of X of length μ' (running from $\langle 1 \rangle$ to X of course) with each factor nilpotent (and possibly trivial). Clearly S_X is finite; by [4] IX.6.4 and IX.6.9 it is also non-empty. If $X \leqslant Y \in \mathbf{X}$ then intersection with X defines a map λ_{YX} of S_Y into S_X and $\{\lambda_{YX} : X, Y \in \mathbf{X}$ with $X \leqslant Y\}$ is an inverse system of non-empty finite sets. Therefore its inverse limit is also non-empty (e.g. [5] 1.K.1).

Let $(N_{X,i} : 0 \leqslant i \leqslant \mu') \in S_X$ with $\{(N_{X,i}) : X \in \mathbf{X}\}$ lying in this inverse limit. Set $N_i = \cup_X N_{X,i}$ for each i for $0 \leqslant i \leqslant \mu'$. Then the N_i form a normal series of S of length μ'. Also $X \cap N_i = N_{X,i}$ for each X and i and hence

$$(X \cap N_{i+1})N_i/N_i \simeq (X \cap N_{i+1})/(X \cap N_i) = N_{X,i+1}/N_{X,i}.$$

Thus each factor N_{i+1}/N_i is locally nilpotent. By the Hirsch-Plotkin Theorem we can replace the N_i (and hence also S) by characteristic subgroups of G.

f) Apply e) to $O_{p'}(G)$. With S as in e) clearly $C_S(\phi) = \langle 1 \rangle$ and then $C_{G/S}(\phi) = C_G(\phi)S/S$ by Lemma 2.1. Now apply Part d) to G/S.

□

Proof of Theorems 1 and 2 — In Theorems 1.1 and 1.2 the group G satisfies min-q for every prime q and therefore $\tau(G)$ is (locally soluble)-by-finite by Belyaev's Theorem, see [3] 3.5.15.
Theorem 1.1 is now immediate from Lemma 2.5. It is easy to see that a periodic locally nilpotent group satisfying min-q for every prime q is a hypercentral. Thus Theorem 1.2 follows from Lemma 2.7. □

A special case — In Lemma 2.7 suppose $p = 2$ and $m = 4$. Then in Part b) of the lemma from [8] we know that G is a finite extension a soluble group of derived length at most 3. Thus in Parts c) and d) we obtain that G is a finite extension of a soluble group of derived length at most 4. For Part e) a simple localization argument (simpler

that the argument needed for the proof of Part e) in general) and the finite case yield that for Part e) the group G is soluble of derived length at most 3 (no finite extension needed here).
Finally, therefore, we arrive at the conclusion that in general if $m = 4$ then G is a finite extension of a soluble group of derived length at most 7. I would be surprised if 7 is actually the best value.

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