Advances in Group Theory and Applications
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1 (2016), pp. 131–137
ISSN: 2499-1287
DOI: 10.4399/97888548908179

Maximal Subgroup Containment in Direct Products

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(*Received Jan. 21, 2016; Accepted Feb. 18, 2016 — Communicated by A. Ballester-Bolinches*)

Dedicated to Hermann Heineken

Abstract

Using the main theorem from [1] that characterizes containment of subgroups in a direct product, we provide a characterization of maximal subgroups contained in a direct product. We also provide an example of our main theorem to a maximal subgroup in $A_4 \times A_4$.

Mathematics Subject Classification (2010): 20D40, 20E15

Keywords: maximal subgroups; non-supersolvable groups; direct products

1 Introduction

In [1], we specify exact conditions on two subgroups U_1 and U_2 of the direct product of two groups A and B that characterize when $U_2 \leqslant U_1$. As applications, we calculated and presented the subgroup lattice of $Q_8 \times Q_8$, where Q_8 is the quaternion group of order 8. The second author has several other similar examples, as well as more details of technicalities, in her dissertation [2]. The groups used are supersolvable, even nilpotent, which made order of subgroup sufficient for determining maximality of one subgroup in another.

The article [4] tackled finding the maximal subgroups of a direct product, but to apply the ideas of [1] to get the subgroup lattice of a non-supersolvable group we want a characterization of the maximal subgroups of a subgroup of a direct product. That is what the main theorem in this article accomplishes.

We conclude by applying our main theorem to compute the second maximal subgroups in $A_4 \times A_4$ that are maximal in a maximal subgroup of $A_4 \times A_4$ that is of diagonal type.

2 Preliminaries

Basic to our results, as well as those in [4], is what is sometimes called Goursat's Theorem, which gives full description of the subgroups of a direct product of two groups.

For groups A, B and $U \leqslant A \times B$, we use $I = \pi_A(U)$, $L = \pi_B(U)$, where π denotes the projection to the respective coordinate groups, and $J = A \cap U$, $K = U \cap B$ (we think of direct products internally for the most part). Then there is an isomorphism $\sigma : I/J \longrightarrow L/K$ and U is completely determined by $U = \{ab | (aJ)^\sigma = bK\}$.

We associate U to the triple $(I/J, L/K, \sigma)$. We will be concerned with two subgroups U_1 and U_2 of $A \times B$ and their associated triples $(I_i/J_i, L_i/K_i, \sigma_i)$, for $i = 1, 2$.

Lemma 2.1 *Let* $U_1, U_2 \leqslant G = A \times B$. *If* $U_2 < \cdot\, U_1$ *and* $J_1 \times K_1 \nleq U_2$, *then* $I_2 J_1 = I_1$ *and* $L_2 K_1 = L_1$.

Proof — Since $U_2 < \cdot\, U_1$, $U_1 = U_2(J_1 \times K_1)$. Then

$$I_2 J_1 = \pi_A(U_2(J_1 \times K_1)) = \pi_A(U_1) = I_1.$$

Similarly, one can prove $L_2 K_1 = L_1$. □

Theorem 2.2 ([1], Theorem 2.4) *Let* $U_1, U_2 \leqslant G = A \times B$ *with corresponding triples* $(I_n/J_n, L_n/K_n, \sigma_n)$ *for* $n = 1, 2$. *Then* $U_2 \leqslant U_1$ *if and only if for* $X \in \{I, J, K, L\}$, $X_2 \leqslant X_1$, $(I_2 J_1/J_1)^{\sigma_1} = L_2 K_1/K_1$, $((I_2 \cap J_1)/J_2)^{\sigma_2} = (L_2 \cap K_1)/K_2$ *and* $\overline{\sigma}_1 \theta_2 = \theta_1 \overline{\sigma}_2$, *where for* $n = 1, 2$, θ_n, $\overline{\sigma}_n$ *are as defined in* [1].

Observation It is routine to verify (and we use frequently) that if $U_2 \leqslant U_1$ (in the notation of 2.2) and $X_1 = X_2$, for all $X \in \{I, J, K, L\}$, then $U_2 = U_1$. That is, $\sigma_1 = \sigma_2$.

Lemma 2.3 *If* $\phi : X \to Y$ *is an epimorphism with* $M <\cdot X$*, then* $\phi(M) = Y$ *or* $\phi(M) < \cdot Y$.

PROOF — Suppose $\phi(M) < R$. Then $M < \phi^{-1}(R)$. Since ϕ is onto, $\phi^{-1}(R) \neq X$. Therefore, $\phi^{-1}(R) = M$. □

One generally thinks of subgroup lattices of finite groups [3], but our results need the following property.

Property 2.4 *If* $M \leqslant N$ *are subgroups of a group* G*, then* $G/M \simeq G/N$ *if and only if* $M = N$.

Of course finite groups have this property and are predominantly in our mind, but should one want a bit more generality, only Property 2.4 is required to establish our results.

3 The Main Theorem

Assume all groups and subgroups thereof satisfy Property 2.4.

Theorem 3.1 *Suppose* $U_n \leqslant A \times B$ *with* U_n *corresponding to the triple* $(I_n/J_n, L_n/K_n, \sigma_n)$*, where* $n = 1, 2$*. Then* $U_2 < \cdot U_1$ *if and only if*

(i) $U_2 \leqslant U_1$*, and*

(ii) *(I.)* *If* $J_1 \times K_1 \leqslant U_2$*, then* $I_2 < \cdot I_1$.

(II.) *If* $J_1 \times K_1 \nleqslant U_2$*, then either*

(a) $K_1 \leqslant U_2$ *and consequently* $I_2 < \cdot I_1$ *and* $L_2 = L_1$*, or*

(b) $J_1 \leqslant U_2$ *and consequently* $L_2 < \cdot L_1$ *and* $I_2 = I_1$*, or*

(c) $J_1 \nleqslant U_2$ *and* $K_1 \nleqslant U_2$ *and consequently* $I_2 = I_1$, $L_2 = L_1$*, and* J_1/J_2 *is a chief factor of* I_1.

PROOF — "$\Longrightarrow$"
In case *(I.)*, observe that $J_1 = J_2$ and $K_1 = K_2$. Now, $U_1/(J_1 \times K_1) \simeq I_1/J_1$ via projections and the image of $U_2/(J_1 \times K_1)$ is I_2/J_1. Hence, $I_2 < \cdot I_1$. Consequently, $L_2 < \cdot L_1$. It is not possible for $I_2 = I_1$ and $L_2 = L_1$ for this would mean $U_2 = U_1$, contradicting $U_2 < \cdot U_1$.

For *(II.)(a)*, note $K_1 = K_1 \cap U_2 = (A \cap U_1) \cap U_2 = A \cap U_2 = K_2$ and from $U_2 \leqslant U_1$, we know

$$((I_2 \cap J_1)/J_2)^{\sigma_2} = (L_2 \cap K_1)/K_2 = K_2/K_2 = 1.$$

Therefore, $\sigma_2 : I_2/J_2 \longrightarrow L_2/K_2$ is an isomorphism, and hence $(I_2 \cap J_1)/J_2 = 1$. Thus, $I_2 \cap J_1 = J_2$. From 2.1, we know $I_2J_1 = J_1$. Thus,

$$L_1/K_1 \simeq I_1/J_1 = I_2J_1/J_1 \simeq I_2/(I_2 \cap J_1) = I_2/J_2 \simeq L_2/K_2.$$

If $L_2 < L_1$, then $L_2/K_2 < L_1/K_1 \simeq L_2/K_2$. So, $L_2 = L_1$.

If $I_1 = I_2$, then $J_1 = J_2$. This means $U_2 = U_1$, contrary to $U_2 < \cdot\, U_1$. Then, by 2.3, $I_2 < \cdot\, I_1$.

(II.)(b) can be proved analogously with respect to factors.

For *(II.)(c)*, we know $U_2 \leqslant U_2K_1 \leqslant U_1$ and $U_2K_1 \neq U_2$ since $K_1 \nleqslant U_2$. So, $U_2K_1 = U_1$. Then $I_1 = \pi_A(U_1) = \pi_A(U_2K_1) = I_2$. Similarly, we can consider $U_2 < U_2J_1 \leqslant U_1$ and use $J_1 \nleqslant U_2$ to get $L_1 = L_2$.

Observe that $U_1 = U_2J_1$ and $U_2 \cap J_1 = U_2 \cap (U_1 \cap A) = U_2 \cap A = J_2$. To show J_1/J_2 is a chief factor of I_1, suppose N is I_1-invariant with $J_2 < N < J_1$ and consider $U_2 \leqslant U_2N \leqslant U_1$.

If $U_2 = U_2N$, then $N \leqslant U_2$. Since $N \leqslant A$, $N \leqslant U_2 \cap A = J_2$, which is a contradiction to our assumption.

If $U_1 = U_2N$, then $J_1 = J_1 \cap U_2N = (J_1 \cap U_2)N = J_2N = N$, contrary to our assumption.

Therefore, there does not exist such an N, and J_1/J_2 is a chief factor of I_1. Similarly, K_1/K_2 is a chief factor of L_1.

"$\Longleftarrow$"

Case 1 Assume (i) and (ii)(I.) holds. Note $J_1 \times K_1 \leqslant U_2$ and $I_2 < \cdot\, I_1$.

Then $\pi_A : U_1/(J_1 \times K_1) \to I_1/J_1$ is an isomorphism, and $\pi_A^{-1}(I_2/J_1) = U_2/(J_1 \times K_1)$. By 2.3, $U_2/(J_1 \times K_1) < \cdot\, U_1/(J_1 \times K_1)$. Therefore, by correspondence, $U_2 < \cdot\, U_1$.

Case 2 (subcase 1) Assume (i) and (II.)(a) holds. Note $J_1 \times K_1 \nleqslant U_2$, $K_1 \leqslant U_2$, $I_2 < \cdot\, I_1$ and $L_2 = L_1$.

$$K_2 = U_2 \cap B = U_2 \cap U_1 \cap B = U_2 \cap K_1 = K_1.$$

Let $U_2 \leqslant W \leqslant U_1$, where $W = (I/J, L/K, \sigma)$. Then $L = L_1 = L_2$, $K = K_1$, $I_2 \leqslant I \leqslant I_1$, and $J_2 \leqslant J \leqslant J_1$. Since $I_2 < \cdot\, I_1$, $I = I_2$ or $I = I_1$.

If $I = I_1$, then $J = J_1$ since

$$I_1/J = I/J \simeq L/K = L_1/K_1 \simeq I_1/J_1.$$

Hence, $W = U_1$.

If $I = I_2$, then $J = J_2$ since

$$I_2/J = I/J \simeq L/K = L_1/K_1 = L_2/K_2 \simeq I_2/J_2.$$

Hence, $W = U_2$. Therefore, $U_2 < \cdot\, U_1$.

Case 2 (subcase 2) Assume (i) and (II.)(b) holds. This proof is analogous, with respect to factors, to the proof of subcase 1.

Case 2 (subcase 3) Assume (i) and (II.)(c) holds. Note $J_1 \nleq U_2$, $K_1 \nleq U_2$, $I_2 = I_1$, $L_2 = L_1$, and J_1/J_2 is an I_1-chief factor. Let $U_2 \leqslant W \leqslant U_1$, where $W = (I/J, L/K, \sigma)$. Then $I = I_1 = I_2$ and $L = L_1 = L_2$. Thus, $J \lhd I = I_1$ and $J = J_1$ or $J = J_2$.

If $J = J_1$, then

$$L_1/K = L/K \simeq I/J = I_1/J_1 \simeq L_1/K_1.$$

Since $K \leqslant K_1$, $K = K_1$. Hence, $W = U_1$.

If $J = J_2$, then

$$L_2/K = L/K \simeq I/J = I_2/J_2 \simeq L_2/K_2.$$

Since $K_2 \leqslant K$, $K = K_2$. Hence, $W = U_2$. Therefore, $U_2 < \cdot\, U_1$. □

4 An Example

To see Theorem 3.1 at work, we will consider the direct product $A_4 \times A_4$, where A_4 is the alternating group of order 12. There are 216 subgroups of $A_4 \times A_4$, which include the trivial subgroup, 12 subgroups of order 2, 43 subgroups of order 3, 35 subgroups of order 4, 24 subgroups of order 6, 15 subgroups of order 8, 16 subgroups of order 9, 50 subgroups of order 12, 1 subgroup of order 16, 6 subgroups of order 24, 8 subgroups of order 36, 4 subgroups of order 48, and the group itself. Establishing the notation for the subgroups of A_4, we denote the Klein 4-group as V, and the four cyclic groups of order 3 as $F_i = \langle f_i \rangle$, where $1 \leqslant i \leqslant 4$.

Let U_1 be the subgroup of order 48 corresponding to the triple

$$(A_4/V, A_4/V, \mathrm{id}).$$

Using Theorem 3.1, (II.)(c), $U_1 < \cdot A_4 \times A_4$. To determine the maximal subgroups, U_2, contained in U_1,we need to verify Theorem 3.1 (i) and (ii).

Observe that verifying (i) for U_2 is routine, and so we proceed to verify (ii). For U_1, note $J_1 = K_1 = V$, and $I_1 = L_1 = A_4$.

If $V \times V \leqslant U_2$ and $I_2 < \cdot A_4$, then $I_2 = V$. So, (ii)(I.) gives one maximal subgroup that corresponds to the triple $(V/V, V/V, id)$, which is the direct product $V \times V$.

If $V \times V \nleqslant U_2$, $V \leqslant U_2$, $I_2 < \cdot A_4$ and $L_2 = A_4$, then $I_2 = F_i$ and $K_2 = V$. So, (ii)(II.)(a) gives 4 maximal subgroups that correspond to the triples $(F_i/1, A_4/V, id)$, which is $(1 \times V)\langle(f_i, f_i)\rangle$.
Analogously, with respect to factors, (ii)(II.)(b) gives 4 maximal subgroups that correspond to the triples $(A_4/V, F_i/1, id)$, which is $(V \times 1)\langle(f_i, f_i)\rangle$.

If $V \times V \nleqslant U_2$, $J_1 = K_1 = V \nleqslant U_2$, $I_2 = A_4$, $L_2 = A_4$, and V/J_2 is an A_4-chief factor, then $J_2 = 1 = K_2$. So, (ii)(II.)(c) gives 12 maximal subgroups that correspond to the triples $(A_4/1, A_4/1, \tau_a)$, where τ_a, $a \in A_4$, is the inner automorphism induced by a. More specifically, these subgroups are diagonal subgroups of $A_4 \times A_4$. In order to have set containment, a must be an even permutation.

Therefore, (ii) is satisfied, and by Theorem 3.1, U_1 contains 21 maximal subgroups, including 1 of order 16 and 20 of order 12.

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