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Groups with Boundedly Černikov Conjugacy Classes *

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Abstract

A relevant theorem of B.H. Neumann states that if a group G has boundedly finite conjugacy classes, then its commutator subgroup G' is finite. This result has been generalized in [1], where it is proved in particular that if the orbits of a group G under the action of G' by conjugation are boundedly finite, then the subgroup $\gamma_3(G)$ has boundely finite order. The aim of this paper is to prove a corresponding statement when boundedly finite orbits are replaced by boundedly Černikov orbits.

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1 Introduction

A group G is said to be a CC-*group*, or to have *Černikov conjugacy classes*, if $G/C_G(\langle g\rangle^G)$ is a Černikov group for each element g of G.

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Groups with such a property were introduced as a natural generalization of FC-groups by Polovickiĭ in [8], where he proved in particular that a group G is a CC-group if and only if for every element g of G the factor group $G/C_G(\langle g\rangle^G)$ is periodic and $[g,G]$ is a Černikov group. It follows in particular that the commutator subgroup G' of any CC-group G is periodic and it is covered by Černikov G-invariant subgroups.

Several authors have investigated the structure of groups with the CC-property, with the purpose of extending known results concerning FC-groups to this more general situation (see for instance [2],[3],[4],[7]). A relevant theorem of B.H. Neumann [6] states that if a group G has boundedly finite conjugacy classes (i.e. it is a BFC-*group*), then its commutator subgroup G' is finite. This result has been generalized in [1], where it is proved in particular that if in a group G the orbits under the action of G' by conjugation have finite bounded orders, then the subgroup $\gamma_3(G)$ is finite and has bounded order. The aim of this paper is to obtain a corresponding result for groups in which the action by conjugation of the commutator subgroup determines bounded Černikov orbits. The first problem in this context is of course to choose in which sense a CC-group may have *bounded* conjugacy classes.

Any Černikov group Q contains a divisible abelian normal subgroup $J(Q)$ (its *finite residual*, actually), which is the direct product of a finite number $m(Q)$ of groups of Prüfer type, such that the factor group $Q/J(Q)$ is finite, of order $i(Q)$ say. The non-negative integers $m = m(Q)$ and $i = i(Q)$ are invariants of Q, and the pair (m,i) is usually called the *size* of Q. Of course, a Černikov group Q is finite if and only if $m(Q) = 0$. Notice also that $m(Q)$ is the *minimax rank* of the divisible abelian group $J(Q)$, so that both the rank of Q and the number of primes involved in Q are bounded by $m(Q) + i(Q)$.

If h and k are non-negative integers (with $k \neq 0$), a group G is called a BCC-*group of size* (h,k) if for each element g of G the factor group $G/C_G(\langle g\rangle^G)$ is a Černikov group with

$$m\big(G/C_G(\langle g\rangle^G)\big) \leqslant h \quad \text{and} \quad i\big(G/C_G(\langle g\rangle^G)\big) \leqslant k,$$

and h,k are the smallest integers with such a property. Thus a group G has the BFC-property considered by Neumann if and only if it is a BCC-group of size $(0,k)$ for some positive integer k. It was claimed in the statement of the Main Theorem of [5] that every BCC-group has a Černikov commutator subgroup; on the other hand, in the review of that paper Tresch pointed out a serious mistake in the proof of the crucial Lemma 2 (see MR1956638). Afterwards, Tresch himself proposed in [11] a different proof of the same result, but unfortunately also in this case a wrong argument occurred in the last part of the proof. Our first result shows that the above quoted statement is actually true.

Theorem A *Let* G *be a* BCC*-group of size* (h, k)*. Then the commutator subgroup* G' *of* G *is a Černikov group and its size is bounded in terms of* h *and* k*.*

We shall say that a group G is a BCC^2*-group* if there are integers $h \geqslant 0$ and $k > 0$ such that $G/C_G(\langle g\rangle^G)$ is a BCC-group of size at most (h, k) for every element g of G, and (h, k) is the *size* of G if h and k are the smallest integers with such a property. Our second main result corresponds to the above mentioned theorem in [1], when one moves from finite to Černikov orbits under the action by conjugation of the commutator subgroup.

Theorem B *Let* G *be a* BCC^2*-group of size* (h, k)*. Then* $\gamma_3(G) = [G', G]$ *is a Černikov group and its size is bounded in terms of* h *and* k*.*

Our notation is mostly standard and can be found in [9]. In particular, for any group G, we will denote by $\pi(G)$ the set of all prime numbers p for which G contains elements of order p.

2 Proofs

If $\mathfrak{X}$ is any group class, the $\mathfrak{X}C$*-centre* of a group G is the subset consisting of all elements g of G such that the factor group $G/C_G(\langle g\rangle^G)$ belongs to $\mathfrak{X}$, and G is called an $\mathfrak{X}C$*-group* if it coincides with its $\mathfrak{X}C$-centre. In particular, if we choose as $\mathfrak{X}$ the class $\mathfrak{F}$ of finite groups, we obtain the usual definitions of FC-centre and FC-group, respectively; similarly, the choice of $\mathfrak{X}$ as the class $\mathfrak{C}$ of Černikov groups gives rise to the concepts of CC-centre and CC-group, respectively. Notice here that the $\mathfrak{X}C$-centre of a group G need not in general be a subgroup; this is for instance the case if $\mathfrak{X}$ is the class $\mathfrak{F} \cup \mathfrak{A}$ of all groups which are either finite or abelian. On the other hand, it is straightforward to show that the $\mathfrak{X}C$-centre of any group is a subgroup, provided that the group class $\mathfrak{X}$ is closed with respect to subgroups, homomorphic images and direct products of finitely many factors.

Lemma 1 *Let $\mathfrak{X}$ be a group class, and let G be a group such that the factor group $G/C_G(\langle g\rangle^G)$ has the $\mathfrak{X}C$-property for each element g of G. If x and y are elements of G, then $G/C_G(\langle [x,y]\rangle^G)$ is isomorphic to a section of a direct product of two $\mathfrak{X}$-groups.*

Proof — Since $G/C_G(\langle x\rangle^G)$ and $G/C_G(\langle y\rangle^G)$ are $\mathfrak{X}C$-groups, there exist normal subgroups M and N of G such that G/M and G/N belong to $\mathfrak{X}$ with

$$[M, \langle y\rangle^G] \leqslant C_G(\langle x\rangle^G) \quad \text{and} \quad [N, \langle x\rangle^G] \leqslant C_G(\langle y\rangle^G).$$

Then

$$[M \cap N, \langle x\rangle^G, \langle y\rangle^G] = [\langle y\rangle^G, M \cap N, \langle x\rangle^G] = \{1\}$$

and so it follows from the Three Subgroup Lemma that

$$[\langle x\rangle^G, \langle y\rangle^G, M \cap N] = \{1\}.$$

In particular, $M \cap N$ is contained in the centralizer $C_G(\langle [x,y]\rangle^G)$, and hence the group $G/C_G(\langle [x,y]\rangle^G)$ is isomorphic to a section of the direct product $G/M \times G/N$. □

The above statement has the following obvious consequence, where r is any positive integer and $\mathfrak{R}_r$ denotes the class consisting of all groups of rank at most r.

Corollary 2 *Let r be a positive integer, and let G be a group such that $G/C_G(\langle g\rangle^G)$ is an $\mathfrak{R}_r C$-group for each element g of G. Then the factor group $G/C_G(\langle [x,y]\rangle^G)$ has rank at most $2r$ for all elements x and y of G.*

It is well-known that any abelian-by-finite FC-group is finite over its centre, and a corresponding easy result holds for groups with the CC-property.

Lemma 3 *Let G be a CC-group containing an abelian normal subgroup A of finite index m. Then $G/Z(G)$ is a Černikov group. Moreover, if the group $G/C_G(\langle g\rangle^G)$ has minimax rank at most h for each $g \in G$, then the minimax rank of $G/Z(G)$ is at most mh.*

Proof — Let $\{g_1, \dots, g_m\}$ be a transversal to A in G. The intersection

$$C = A \cap \Big(\bigcap_{i=1}^{m} C_G(\langle g_i \rangle^G) \Big)$$

is obviously contained in the centre of G, and the factor group G/C embeds into the direct product

$$G/A \times \Big(\mathop{\mathrm{Dr}}_{i=1}^{m} G/C_G(\langle g_i \rangle^G) \Big).$$

Therefore $G/Z(G)$ is a Černikov group, and its minimax rank is at most mh if the second condition is satisfied. □

Lemma 4 *Let* G *be a group such that* $G/Z(G)$ *is a Černikov group of size* (h, k). *Then* G' *is a Černikov group and its size is bounded by a function* $c_1 = c_1(h, k)$.

Proof — Let $J/Z(G)$ be the finite residual of $G/Z(G)$, so that $J/Z(G)$ is the direct product of h Prüfer subgroups and G/J has order k. Since $J/Z(G)$ is a periodic divisible abelian group and J' is a homomorphic image of the tensor product $J/Z(G) \otimes J/Z(G)$, it follows that J is abelian. Let $\{g_1, \dots, g_k\}$ be a transversal to J in G. Then

$$[J, G] = \big\langle [J, g_1], \dots, [J, g_k] \big\rangle = [J, g_1] \cdot \ldots \cdot [J, g_k].$$

For each $i = 1, \dots, k$ the subgroup $[J, g_i] \simeq J/C_J(g_i)$ is a homomorphic image of $J/Z(G)$, and hence it is a direct product of at most h Prüfer subgroups. It follows that $[J, G]$ is a direct product of at most hk Prüfer subgroups. Put $\overline{G} = G/[J, G]$. Then $\overline{G}/Z(\overline{G})$ is finite of order at most k, so that the quantitative version of Schur's theorem yields that the commutator subgroup $\overline{G}'$ is finite and its order is bounded by a function of k (see for instance [9] Part 1, p.103). Therefore G' is a Černikov group and its size is bounded in terms of h and k. □

Notice that if we use the bound for the order of the commutator subgroup of a central-by-finite group obtained by Wiegold in [12], the function c_1 in the above statement can be chosen as

$$c_1(h, k) = (hk, k^{\frac{1}{2}\log_p k - 1}),$$

where p is the least prime number dividing k.

Lemma 5 *Let G be a metabelian group, and let g be any element of G. Then:*

(a) *the subgroup $[g, G']$ is normal in G and $\langle g\rangle^{G'} = \langle g\rangle[g, G']$;*

(b) *the subgroup $C_{G'}(g)$ is normal in G and $G'/C_{G'}(g) \simeq [g, G']$.*

Proof — Since G' is abelian, we have

$$[g, G'] = [\langle g, G'\rangle, G'] \quad \text{and} \quad C_{G'}(g) = C_{G'}(\langle g, G'\rangle),$$

and so the subgroups $[g, G']$ and $C_{G'}(g)$ are normal in G. Moreover, the groups $G'/C_{G'}(g)$ and $[g, G']$ are isomorphic, because

$$[g, y_1y_2] = [g, y_1][g, y_2]$$

for all elements y_1, y_2 of G'. □

Lemma 6 *Let r and s be positive integers, and let G be a finite metabelian group such that $G/C_G(\langle g\rangle^G)$ has the $\mathfrak{R}_r C$-property and $G'/C_{G'}(g)$ has rank at most s for each element g of G. Then the rank of the subgroup $\gamma_3(G)$ is bounded by $(2rs+1)s$.*

Proof — Put $A = \gamma_3(G)$, and let p be a prime number such that the rank of A coincides with that of its p-component. Then A and $A/A^pO_{p'}(A)$ have the same rank, and so the replacement of G by $G/A^pO_{p'}(A)$ allows us to assume that A has exponent p. Let x be an element of G such that $G'/C_{G'}(x)$ has maximal rank s. Since $G'/C_{G'}(x) \simeq [x, G']$ has exponent p, there exist s commutators $a_1, \dots, a_s$ such that

$$G'/C_{G'}(x) = \langle a_1C_{G'}(x), \dots, a_sC_{G'}(x)\rangle,$$

and so

$$[x, G'] = [x, \langle a_1, \dots, a_s\rangle].$$

Write $C = C_G(\langle a_1, \dots, a_s\rangle^G)$. It follows from Corollary 2 that the group $G/C_G(\langle a_i\rangle^G)$ has rank at most $2r$ for all i, and hence G/C has rank at most $2rs$. In particular, G/C admits a set of generators $\{g_1C, \dots, g_tC\}$ with $t \leqslant 2rs$.

Assume for a contradiction that $[C, G']$ is not contained in $[x, G']$, so that there exist elements c of C and y of G' such that $[c, y]$ does not belong to $[x, G']$. If a is any element of $\langle a_1, \dots, a_s\rangle$, we have

$$[cx, a] = [c, a]^x [x, a] = [x, a]$$

and hence $[x, G']$ is contained in $[cx, G']$. On the other hand,

$$[cx, y] = [c, y]^x [x, y]$$

is not in $[x, G']$, so that $[x, G']$ is properly contained in $[cx, G']$, which is impossible by the choice of x. This contradiction shows that $[C, G']$ is a subgroup of $[x, G']$, and so it has rank at most s. If $\overline{G} = G/[C, G']$, we have $[\overline{C}, \overline{G}'] = \{1\}$, so that

$$\gamma_3(\overline{G}) = \left\langle [\overline{g}_1, \overline{G}'], \dots, [\overline{g}_t, \overline{G}'] \right\rangle$$

and hence $\gamma_3(\overline{G})$ has rank at most $ts \leqslant 2rs^2$. Therefore the rank of $A = \gamma_3(G)$ is bounded by $(2rs + 1)s$. □

Lemma 7 *Let G be a group and let $N = \langle X\rangle$ be a normal subgroup of G. If $[g, N]$ is a Černikov subgroup of $Z(N)$ and $|\pi([g, N])| \leqslant t$ for every element g of G, then there exists a finite subset Y of X of order at most t such that the order of $\pi([C_G(\langle Y\rangle^G), N])$ is bounded by t. In particular, if $[G, N]$ has finite rank, then $[C_G(\langle Y\rangle^G), N]$ is a Černikov group.*

Proof — Let z be an element of G such that the set

$$\pi([z, N]) = \{p_1, \dots, p_s\}$$

has largest order s. Clearly,

$$[z, N] = \left\langle [z, x] \mid x \in X \right\rangle^N = \left\langle [z, x] \mid x \in X \right\rangle,$$

and so for each $i = 1, \dots, s$ there exists an element x_i of X such that the commutator $[z, x_i]$ has order divisible by p_i. Put $Y = \{x_1, \dots, x_s\}$ and $C = C_G(\langle Y\rangle^G)$, so that $\pi([z, \langle Y\rangle]) = \{p_1, \dots, p_s\}$.

Assume for a contradiction that the set $\pi([C, N])$ contains a prime number q which is not in $\{p_1, \dots, p_s\}$. Since

$$[C, N] = \left\langle [c, N] \mid c \in C \right\rangle,$$

there exist elements a of N and b of C and such that $[b, a]$ has order q, and clearly q divides the order of $[bz, a] = [b, a]^z[z, a]$. Moreover, $[bz, \langle Y\rangle] = [z, \langle Y\rangle]$ so that

$$\{p_1, \dots, p_s\} = \pi([z, \langle Y\rangle]) = \pi([bz, \langle Y\rangle]) \subseteq \pi([bz, N])$$

and hence the set $\pi([bz, N])$ contains more than s elements, which is impossible by the choice of z. Therefore $\pi([C, N])$ is contained in $\{p_1, \dots, p_s\}$, and so it has order at most t. □

We will also use the well-known fact that every finitely generated metabelian group is residually finite (see for instance [9] Part 2, Theorem 9.51).

Lemma 8 *Let G be a metabelian BCC^2-group of size (h, k) such that for each element g of G the subgroup $[g, G']$ is Černikov of rank at most s. Then the subgroup $\gamma_3(G)$ has finite rank bounded by a function of h,k and s.*

Proof — Let $E = \langle g_1, \dots, g_t\rangle$ be any finitely generated subgroup of G. Then

$$\begin{aligned}\gamma_3(E) &= \big[\{g_1, \dots, g_t\}, E'\big]^E = [g_1, E']^E \cdot \ldots \cdot [g_t, E']^E\\ &= [g_1, E'] \cdot \ldots \cdot [g_t, E']\end{aligned}$$

is a Černikov group, and so it is even finite, because E is residually finite. Thus there exists a normal subgroup of finite index K of E such that $\gamma_3(E) \cap K = \{1\}$. The finite group E/K obviously satisfies the assumptions of Lemma 6 with $r = h + k$, and hence the rank of $\gamma_3(E) \simeq \gamma_3(E/K)$ is bounded by $w = \big(2(h + k)s + 1\big)s$. Therefore $\gamma_3(G)$ has rank at most w. □

Lemma 9 *Let G be a metabelian BCC^2-group of size (h,k) such that $\gamma_3(G)$ has finite rank and for each element g of G the subgroup $[g, G']$ is Černikov with $|\pi\big([g, G']\big)| \leqslant t$. Then $\gamma_3(G)$ is a Černikov group and the order of the set $\pi\big(\gamma_3(G)\big)$ is bounded by a function of h,k and t.*

Proof — An application of Lemma 7 for $N = G'$ yields that there exists a subgroup E of G generated by $m \leqslant t$ commutators $u_1, \dots, u_m$ such that the set $\pi\big([C_G(E^G),G']\big)$ has order at most t, and $[C_G(E^G), G']$ is a Černikov group since $\gamma_3(G)$ has finite rank. Thus it is enough to prove the statement for the factor group $G/[C_G(E^G), G']$, and hence without loss of generality it can be assumed that $G/C_G(G')$ is a homomorphic image of $G/C_G(E^G)$. Since the factor group $G/C_G(E^G)$ can be embedded into the direct product

$$\operatorname*{Dr}_{i=1}^{m}\big(G/C_G(\langle u_i\rangle^G)\big),$$

we have by Lemma 1 that $G/C_G(G')$ is a Černikov group of rank at most $2(h+k)t$ and $|\pi\big(G/C_G(G')\big)| \leqslant 2(h+k)t$.

Put $C = C_G(G')$, and let J/C be the finite residual of G/C. Consider in J/C a subgroup P/C of type p^∞, and for each positive integer n let x_n be an element of P such that $\langle x_n, C\rangle/C$ is the unique subgroup of order p^n of P/C. Then

$$[x_1, G'] \leqslant [x_2, G'] \leqslant \ldots \leqslant [x_n, G'] \leqslant \ldots$$

and

$$[P, G'] = \bigcup_{n\in\mathbb{N}} [x_n, G'].$$

As the orders of the sets $\pi\big([x_n, G']\big)$ are bounded by t, there exists a positive integer s such that $\pi\big([P, G']\big) = \pi\big([x_s, G']\big)$ and hence $|\pi\big([P, G']\big)| \leqslant t$; moreover, $[P, G']$ is a Černikov group because $\gamma_3(G)$ has finite rank. Since J/C is the direct product of at most $2ht$ Prüfer subgroups, it follows that also $[J, G']$ is a Černikov group and $|\pi\big([J, G']\big)| \leqslant 2ht^2$. Again, the group G can be replaced by $G/[J, G']$, and so we may suppose that the abelian group G/C is finite. Then

$$G/C = \langle g_1C\rangle \times \ldots \times \langle g_jC\rangle,$$

where $j \leqslant 2(h+k)^2t^2$, and hence

$$\gamma_3(G) = [g_1, G'] \cdot \ldots \cdot [g_j, G']$$

is a Černikov group and $|\pi\big(\gamma_3(G)\big)| \leqslant 2(h+k)^2t^3$. Taking in mind all the reductions that we have done, we obtain that in the original group G the subgroup $\gamma_3(G)$ is Černikov and the order of the set $\pi\big(\gamma_3(G)\big)$ is bounded by $t + 2ht^2 + 2(h+k)^2t^3$. The statement is proved. □

Proof of Theorem A — Let D be the subgroup generated by all its periodic abelian divisible normal subgroups and put $\widehat{G} = G/D$. Since for each element g of G the normal subgroup $[g, G]$ is a Černikov group, it follows that $[\widehat{g}, \widehat{G}]$ is finite, and so $\widehat{G}$ is an FC-group. Moreover, the factor group $\widehat{G}/C_{\widehat{G}}(\langle \widehat{g} \rangle^{\widehat{G}})$ is a finite homomorphic image of $G/C_G(\langle g \rangle^G)$, and hence it has order at most k. Therefore $\widehat{G}$ is a BFC-group, and so Neumann's theorem yields that $\widehat{G}' = G'D/D$ is finite and its order m is bounded in terms of k.

On the other hand, it is well-known that the subgroup D is abelian (see for instance [9] Part 1, Lemma 4.46), so that the CC-group $G'D$ contains an abelian subgroup of index m. It follows from Lemma 3 that $G'D/Z(G'D)$ is a Černikov group of minimax rank bounded by mh, and so in terms of h and k. Moreover, $DZ(G'D)/Z(G'D)$ is contained in the finite residual of $G'D/Z(G'D)$, and hence the size of $G'D/Z(G'D)$ is bounded by (mh, m). An application of Lemma 4 yields now that $(G'D)'$ is a Černikov group whose size is bounded by a function of mh and m, and so in terms of h and k. Thus it is enough to prove the statement for the factor group $G/(G'D)'$, and hence it can be assumed without loss of generality that G is metabelian.

It follows now from Lemma 5 that for each element g of G the subgroup $[g, G']$ is isomorphic to $G'/C_{G'}(g)$, and so it is a Černikov group of rank at most $h + k$. It follows from Lemma 8 that $\gamma_3(G)$ has finite rank bounded by a function of h and k. Clearly, $\pi\big([g, G']\big)$ has order at most $h + k$ for each element g of G, and so it follows from Lemma 9 that $\gamma_3(G)$ is a Černikov group and $|\pi\big(\gamma_3(G)\big)|$ is bounded by a function of h and k. Thus also the minimax rank of $\gamma_3(G)$ is bounded in terms of h and k and hence, in order to prove that G' is a Černikov group whose minimax rank is bounded by a function of h and k, we may replace G by $G/\gamma_3(G)$ and suppose that G is nilpotent of class 2.

For each element g of G the groups $G/C_G(g)$ and $[g, G]$ are isomorphic, so that $\langle g \rangle^G = \langle g \rangle[g, G]$ has rank at most $h + k + 1$, and it follows from a result of H. Smith [10] that G' has finite rank ν, bounded in terms of h and k. Of course, $|\pi([g, G])| \leqslant h + k$ for each g, and hence an application of Lemma 7 for $N = G$ yields that G contains a subgroup E, generated by at most $h + k$ elements, such that the set $\pi([C_G(E^G), G])$ is finite and has order at most $h + k$. Since G' is periodic, we have that $[C_G(E^G), G]$ is a Černikov group of minimax rank at most $(h + k)\nu$. Moreover, $G/C_G(E^G)$ is a Černikov group and $\pi = \pi\big(G/C_G(E^G)\big)$ has order at most $h(h + k)$. Put $C = C_G(E^G)$ and $G^* = G/[C, G]$. Then C^* is contained in $Z(G^*)$, so that $G^*/Z(G^*)$ is a Černikov π-group. Then also the commutator subgroup $(G^*)'$

of G^* is a Černikov π-group and its minimax rank is at most $h(h+k)\nu$. It follows that also in the original group G the commutator subgroup G' is a Černikov group whose minimax rank is bounded in terms of h and k.

Finally, let J be the finite residual of G', and put $\widetilde{G} = G/J$. Then $\widetilde{G}'$ is finite, so that $\widetilde{G}$ is a BFC-group. As G is a BCC-group of size (h, k), it follows that each element of $\widetilde{G}$ has at most k conjugates, and hence the order of $\widetilde{G}' = G'/J$ is bounded by a function of k. Therefore the size of the Černikov group G' is bounded by a function of h and k, and the proof is complete. □

Proof of Theorem B — Let $g \in G$. Since G is a BCC^2-group of size (h, k), the factor group $G/C_G(\langle g\rangle^G)$ has the BCC-property with size (h, k) and so it follows from Theorem A that $G'/C_{G'}(\langle g\rangle^G)$ is a Černikov group whose size is bounded by $f(h, k)$, for a suitable function $f : \mathbb{N}_0^2 \to \mathbb{N}_0^2$. Thus G' is a BCC-group of size $(h', k') = f(h, k)$. A second application of Theorem A yields that G'' is a Černikov of size bounded in terms of h' and k', and so also of h and k, whence we may replace G by the factor group G/G'' and assume that G is metabelian. Then $[g, G'] \simeq G'/C_{G'}(g)$ is a Černikov group of rank at most $h' + k'$ for each element g of G, and so it follows from Lemma 8 that the subgroup $\gamma_3(G)$ has finite rank, again bounded by a function $u = u(h', k')$. Moreover $|\pi([g, G'])| \leqslant h' + k'$ for all g, and hence it follows from Lemma 9 that $\gamma_3(G)$ is a Černikov group and the order of the set $\pi\big(\gamma_3(G)\big)$ is bounded by a function $\nu = \nu(h', k')$. In particular, the largest divisible subgroup J of $\gamma_3(G)$ is the direct product of at most $u \cdot \nu$ Prüfer subgroups.

Put $\overline{G} = G/J$. Then $\gamma_3(\overline{G})$ is finite, and so for each element g of G the subgroup $[\overline{g}, \overline{G}']$ is isomorphic to a finite homomorphic image of $G'/C_{G'}(g)$, and hence its order is at most k'. It follows that each element of $\overline{G}$ has at most k' conjugates under the action of $\overline{G}'$, and so the order of $\gamma_3(G)/J = \gamma_3(\overline{G})$ is bounded by a function of k' (see [1], Theorem 1.2). Therefore the size of $\gamma_3(G)$ can be bounded in terms of h and k, and the proof is complete. □

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