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Groups with Finitely Many Isomorphism Classes of Non-Normal Subgroups *

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Abstract

We study groups in which the non-normal subgroups fall into finitely many isomorphism classes. We prove that a locally generalized radical group with this property is abelian-by-finite and minimax. Here a generalized radical group is a group with an ascending series whose factors are either locally nilpotent or locally finite. We give also a complete description of locally finite groups with finitely many classes of non-normal subgroups.

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1 Introduction

Let G be a group and ν be a subgroup theoretical property. This property can be external to the group as the property of being "an abelian or nilpotent or soluble subgroup", or internal to the group such as the property of being "a normal or subnormal or permutable subgroup". Denote by $\mathcal{L}_\nu(G)$ the family of all subgroups of G having

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the property ν and, respectively, by $\mathcal{L}_{non-\nu}(G)$ the family of all subgroups of G which do not have the property ν. Many authors studied the groups in which the family $\mathcal{L}_{\nu}(G)$ is "very big" in some sense, or the family $\mathcal{L}_{non-\nu}(G)$ is "very small" in some sense, for many natural subgroup properties ν, such as to be a normal, subnormal, permutable, almost normal, infinite, abelian, nilpotent, finitely generated subgroup and many others. But what means "to be small" in infinite group theory? Very often it means "to satisfy some finiteness condition"(see, for example, the survey [6]).

On a group G there are some natural equivalence relations related to its subgroups. One of them is the relation "the subgroups H and K are conjugate". In the paper [12] J. Lennox, F. Menegazzo, H. Smith and J. Wiegold worked with the following relation: "there is an automorphism σ of the group G such that $K = \sigma(H)$". It appears that this relation was very strong.

In this paper we will consider a more general equivalence relation on subgroups. Let G be a group, $\mathcal{M}$ a family of subgroups of the group G and $H, K \in \mathcal{M}$. Then the relation "H is isomorphic to K" is an equivalence relation in $\mathcal{M}$. Denote by $\mathrm{Isom}_{\mathcal{M}}(H)$ the equivalence class of H defined by this relation. Then

$$\mathrm{Isom}_{\mathcal{M}}(H) = \{L \mid L \in \mathcal{M}\ ,\ L \simeq H\}.$$

Choose in every equivalence class one representative and denote the set of all these representatives by $\mathrm{Itype}(\mathcal{M})$. The set $\mathrm{Itype}(\mathcal{M})$ is called the *isomorphism type* of the family $\mathcal{M}$. If $\mathcal{M} = \mathcal{L}(G)$ is the family of all subgroups of G, with $G \neq \{1\}$, then the set $\mathrm{Itype}(\mathcal{L}(G))$ contains at least two elements: G and $\{1\}$. If $\mathrm{Itype}(\mathcal{L}(G))$ contains only these two elements, then clearly G is a group of prime order or G is an infinite cyclic group. In the last case G is isomorphic to each proper non-trivial subgroup. If $|\mathrm{Itype}(\mathcal{L}(G))| = 3$, then the situation is more complicated. In the paper [21] A.Yu. Olshanskii constructed an infinite p-group, whose proper non-trivial subgroups have order p. In [20] A.Yu. Olshanskii constructed a simple torsion-free group, whose proper non-trivial subgroups are cyclic. He used for these examples very complicated constructions, which show that we cannot obtain a full description of groups with $|\mathrm{Itype}(\mathcal{L}(G))| = 3$. But in the universe of generalized soluble groups the description of such groups is not difficult, and also it is possible to obtain information on the structure of groups for which $|\mathrm{Itype}(\mathcal{L}(G))|$ is small. If G is a finite group, then the set $\mathrm{Itype}(\mathcal{M})$ is finite for every family set $\mathcal{M}$ of subgroups of G. Therefore the finiteness of the set $\mathrm{Itype}(\mathcal{L}_{non-\nu}(G))$ is one of the possibilities "to be small" for the family $\mathcal{L}_{non-\nu}(G)$. Thus we come naturally to the following problem: what can be said about the structure of the groups in which the cardinality of the set $\mathrm{Itype}(\mathcal{L}_{non-\nu}(G))$ is finite for some basic property ν of subgroups? One of the basic properties of subgroups is the property "to be an abelian subgroup". The groups in which the family of all non-abelian subgroups has finite isomorphism type were studied in the paper [11]. Under some natural restrictions such groups are minimax and abelian-by-finite. Another important family of subgroups is the family of normal subgroups of G. The behaviour of normal subgroups has an important effect on the structure of a group. There is an enormous array of papers, concerning the groups G in which the family $\mathcal{L}_{norm}(G)$ of normal subgroups "is very big"or the family $\mathcal{L}_{non-norm}(G)$ of all non-normal subgroup "is very small". Of course, if G is an abelian or, more generally, a Dedekind group, then the family $\mathcal{L}_{non-norm}(G)$ is empty. Suppose now that

$$|\mathcal{L}_{non-norm}(G)| = 1.$$

This means that the family $\mathcal{L}_{non-norm}(G)$ is not empty and all non-normal subgroups are isomorphic to some unique subgroup K. If we suppose that K is not cyclic, then each cyclic subgroup of G must be normal. But in this case every subgroup of G is normal, and we obtain a contradiction. This contradiction shows that K is cyclic. It follows that every non-cyclic subgroup of G is normal. Such groups have been described by F.N. Liman in the papers [14] and [15].

In this paper we consider groups in which the family

$$\mathcal{L}_{non-norm}(G)$$

of all non-normal subgroups has finite isomorphism type. We will write $\mathfrak{C}$ this class of groups.

The examples of groups constructed by A.Yu. Olshanskii in [20] and [21] show that a real description of groups in $\mathfrak{C}$ is possible only under some additional restrictions, for example in the universe of generalized soluble groups. Our results are the following ones.

Theorem A *Let G be an infinite locally finite group which is not a Dedekind group. Then G is in the class $\mathfrak{C}$ if and only if $G = P \times A$ where A is a finite Dedekind group, P is a Sylow p-subgroup of G (p a prime), and $\zeta(P)$ includes a Prüfer subgroup D such that P/D is a finite abelian group.*

Theorem B *Let* G *be a locally generalized radical group in* $\mathcal{C}$*. Then either* G *is a Dedekind group, or* G *is abelian-by-finite and minimax.*

Here a locally generalized radical group is a group with an ascending series whose factors are either locally nilpotent or locally finite.

Notice that the direct product of a finite non-Dedekind group with a minimax torsion-free abelian group with infinitely many non-isomorphic subgroups has infinitely many non-normal non-isomorphic subgroups, hence the converse of Theorem B is not generally true.

For the class of locally generalized radical groups we also obtained the following result.

Theorem C *Let* G *be a locally generalized radical group in which the family of all subgroups has finite isomorphic type. Then* G *contains a normal minimax torsion-free abelian subgroup of finite index.*

Finally, we point out that for some other families $\mathcal{M}$ of subgroups of the group G, similar problems have been studied before. For example if $\mathcal{M}$ is the family of the commutator subgroups of all subgroups of G, then groups G with $\mathcal{M}$ finite have been studied by F. de Giovanni and D.J.S. Robinson in [8], as well as by M. Herzog, P. Longobardi, M. Maj in [9]; groups with $\mathcal{M}$ of finite isomorphism type have been investigated by P. Longobardi, M. Maj, D.J.S. Robinson, H. Smith in a series of papers (see [16], [17], and [18]).

Our notation are the usual ones, see for example [13], [22] and [23].

2 The structure of locally finite subgroups

We start our investigation of groups in the class $\mathcal{C}$ with two easy lemmas.

Lemma 2.1 *Suppose that* $G \in \mathcal{C}$*. If* K *is a subgroup of* G*, then* $K \in \mathcal{C}$*.*

Proof — Let K be a subgroup of G. If H is a non-normal subgroup of K, then H is a non-normal subgroup of G. It follows that

$$\mathcal{L}_{non-norm}(K) \subseteq \mathcal{L}_{non-norm}(G).$$

In particular, $\mathrm{Itype}(\mathcal{L}_{non-norm}(K))$ is finite. □

Lemma 2.2 *Suppose that* $G \in \mathcal{C}$. *If* K *is an infinite locally finite subgroup of* G, *then for every finite subgroup* F *of* K *there exists a finite* G*-invariant subgroup* D *of* K *containing* F *such that every subgroup of* K/D *is* G*-invariant. In particular, every infinite locally finite subgroup of* G *is normal in* G *and it is contained in the* FC*-center of* G.

Proof — Write n the isomorphism type of G. If every finite subgroup of K containing F is G-invariant, then every subgroup of K containing F is G-invariant. Suppose that there exists a finite subgroup H_1 of K, $H_1 \geqslant F$, which is not G-invariant. Since K is infinite, K contains a finite subgroup $S > H_1$. It follows that the family

$$\mathcal{L}_1 = \{H \mid H \text{ is a finite subgroup of } K \text{ and } H > H_1\}$$

is not empty. If $\mathcal{L}_1$ contains a finite subgroup H_2 which is not G-invariant, then the subgroups H_1 and H_2 cannot be isomorphic, because $|H_1| < |H_2|$. In this case we consider the family

$$\mathcal{L}_2 = \{H \mid H \text{ is a finite subgroup of } K \text{ and } H > H_2\}.$$

This family is not empty. If $\mathcal{L}_2$ contains a finite subgroup H_3 which is not G-invariant, then the subgroups H_1, H_2 and H_3 cannot be isomorphic. Using similar arguments, we construct a chain of finite subgroups $H_1 < H_2 < \ldots < H_n$ such that every subgroup of the family

$$\mathcal{L}_n = \{H \mid H \text{ is a finite subgroup of } K \text{ and } H > H_n\}$$

is G-invariant. If D is a minimal (by inclusion) element of $\mathcal{L}_n$, then D and every finite subgroup of K containing D are G-invariant. It follows that every subgroup of K/D is G-invariant, in particular, K is normal in G. Now, let x be an arbitrary element of K. Then there is a finite G-invariant subgroup D of K, containing x. The finiteness of D implies that x^G is finite, so that $x \in FC(G)$. □

Corollary 2.3 *Suppose that* $G \in \mathcal{C}$ *and let* P *be an infinite locally finite subgroup of* G. *If* P *is a non-Chernikov subgroup, then every subgroup of* P *is* G*-invariant.*

Proof — It is enough to prove that every finite subgroup of P is G-invariant. Suppose the contrary, let P contain a finite subgroup F which is not G-invariant. By Lemma 2.2 the normal closure L of F is finite. Since P is not a Chernikov group, then P contains an abelian subgroup

$$A = \underset{\lambda \in \Lambda}{\mathrm{Dr}} \langle a_\lambda \rangle,$$

where the set Λ is infinite (see [25]. The intersection $A \cap L$ is finite. Then there is a subset Δ of Λ such that $\Lambda \setminus \Delta$ is finite and

$$L \cap (\underset{\lambda \in \Delta}{\mathrm{Dr}} \langle a_\lambda \rangle) = \langle 1 \rangle.$$

The subgroup $B = \mathrm{Dr}_{\lambda \in \Delta} \langle a_\lambda \rangle$ has finite index in A, in particular, B is infinite. Then Lemma 2.2 implies that B is normal in G. By the choice of B we have $B \cap L = \langle 1 \rangle$, which implies that $[L, B] = \langle 1 \rangle$. Since F is not G-invariant, there exist $x \in F$ and $g \in G$ such that $g^{-1}xg \notin F$. Notice that $g^{-1}xg \in L$. Choose in the subgroup B a countable abelian subgroup $D = \mathrm{Dr}_{n \in \mathbb{N}} \langle d_n \rangle$. Put

$$D_k = \underset{1 \leqslant n \leqslant k}{\mathrm{Dr}} \langle d_n \rangle.$$

Notice that $L \cap (\langle x \rangle D_k) = \langle x \rangle$. It follows that $\langle x \rangle D_k$ is not G-invariant, because $g^{-1}xg \notin \langle x \rangle$. The subgroups $\langle x \rangle D_k$ and $\langle x \rangle D_{k+1}$ cannot be isomorphic, because $|\langle x \rangle D_k| < |\langle x \rangle D_{k+1}|, k \in \mathbb{N}$. Thus we can construct an infinite sequence

$$\langle x \rangle D_1 < \langle x \rangle D_2 < \ldots < \langle x \rangle D_n < \langle x \rangle D_{n+1} < \ldots$$

of finite non G-invariant subgroups, which are pairwise non-isomorphic, and we obtain a contradiction, which proves the result. □

A group G is said to be a *Dedekind group* if every subgroup of G is normal in G. R. Dedekind in his paper [4] studied finite groups whose subgroups are normal. Much later, in the paper [1], R. Baer obtained a full description of such groups, both finite and infinite.

Now we prove our first result on the structure of groups in $\mathfrak{C}$.

Proposition 2.4 *Suppose that $G \in \mathfrak{C}$, and that G contains an infinite locally finite subgroup P which is not Chernikov. Then G is a Dedekind group.*

Proof — By Lemma 2.2, P is normal in G. It is enough to prove that every cyclic subgroup $\langle g \rangle$ of the group G is normal in G. If g has finite order, then the subgroup $P\langle g \rangle$ is locally finite. Clearly $P\langle g \rangle$ is not a Chernikov group. Then Corollary 2.3 shows that every subgroup of $P\langle g\rangle$ is G-invariant. In particular, $\langle g\rangle$ is normal in G. Suppose now that g has infinite order. Since P is not a Chernikov group, P contains an abelian subgroup

$$A = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\,\langle a_\lambda\rangle,$$

where the set Λ is infinite (see [25]). Since A is periodic, $\langle g\rangle \cap A = \langle 1\rangle$. Choose in Λ two infinite subsets Δ and Σ such that

$$\Delta \cup \Sigma = \Lambda \quad \text{and} \quad \Delta \cap \Sigma = \emptyset.$$

In the subgroup $\mathrm{Dr}_{\lambda\in\Delta}\langle a_\lambda\rangle$ (respectively $\mathrm{Dr}_{\lambda\in\Sigma}\langle a_\lambda\rangle$) choose a countable subgroup $B = \mathrm{Dr}_{n\in\mathbb{N}}\langle b_n\rangle$ (respectively $C = \mathrm{Dr}_{n\in\mathbb{N}}\langle c_n\rangle$). By such a choice $B \cap C = \langle 1\rangle$. Using Corollary 2.3 we obtain that every subgroup of B, C is normal in G. Put $B_k = \mathrm{Dr}_{1\leqslant n\leqslant k}\langle b_n\rangle$ (respectively $C_k = \mathrm{Dr}_{1\leqslant n\leqslant k}\langle c_n\rangle$). Clearly the subgroups $\langle g\rangle B_k$ and $\langle g\rangle B_{k+1}$ (respectively $\langle g\rangle C_k$ and $\langle g\rangle C_{k+1}$) cannot be isomorphic, $k \in \mathbb{N}$. Consider now the infinite sequences

$$\langle g\rangle B_1 < \langle g\rangle B_2 < \ldots < \langle g\rangle B_n < \langle g\rangle B_{n+1} < \ldots$$

and

$$\langle g\rangle C_1 < \langle g\rangle C_2 < \ldots < \langle g\rangle C_n < \langle g\rangle C_{n+1} < \ldots$$

The subgroups in each of these sequences are pairwise non-isomorphic. Therefore there exist numbers t and m such that $\langle g\rangle B_t$ and $\langle g\rangle C_m$ are normal in G. Then their intersection

$$\langle g\rangle B_t \cap \langle g\rangle C_m = \langle g\rangle$$

is normal in G. □

Now we study the situation in which G contains an infinite locally finite subgroup which is a Chernikov group.

Lemma 2.5 *Suppose that* $G \in \mathcal{C}$. *If* C *is an infinite Chernikov subgroup of* G, *then every subgroup of the divisible part* D *of* C *is* G*-invariant. In particular,* D *is* G*-invariant. Moreover, If* D *is not a Prüfer group, then* G *is a Dedekind group.*

Proof — Let P be an arbitrary Prüfer p-subgroup of D. Then

$$P = \langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N}\rangle.$$

The subgroups

$$\langle a_1\rangle < \langle a_2\rangle < \ldots < \langle a_n\rangle < \langle a_{n+1}\rangle < \ldots$$

are pairwise non-isomorphic. It follows that there is a number k such that the subgroup $\langle a_n\rangle$ is G-invariant for each $n \geqslant k$. The fact that every subgroup of a cyclic group is characteristic implies that each subgroup $\langle a_m\rangle$ is G-invariant, $m \in \mathbb{N}$. Since $P = \bigcup_{m\in\mathbb{N}}\langle a_m\rangle$, the subgroup P is G-invariant. Now let d be an arbitrary element of D. Since D is divisible, there exists a Prüfer p-subgroup S such that $d \in S$. By the previous remarks S is G-invariant. But every subgroup of a Prüfer p-subgroup is characteristic, thus it follows that $\langle d\rangle$ is G-invariant. The fact that every cyclic subgroup of D is G-invariant implies that each subgroup of D is G-invariant.

Now suppose that D is not a Prüfer group. We prove that every cyclic subgroup $\langle g\rangle$ of the group G is normal in G.

Assume first that g has finite order. Without loss of generality we may suppose that g is a p-element for some prime p. Then the intersection $\langle g\rangle \cap D$ is a finite cyclic p-subgroup. Since D is divisible, there is a Prüfer subgroup A of D, containing $\langle g\rangle \cap D$. Let

$$A = \langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N}\rangle.$$

Let t be the positive integer such that $\langle g\rangle \cap D = \langle a_t\rangle$. Then every subgroup $\langle a_n\rangle$ is G-invariant. Consider the following sequence of subgroups

$$\langle g\rangle = \langle a_t\rangle\langle g\rangle < \langle a_{t+1}\rangle\langle g\rangle < \ldots < \langle a_{t+n}\rangle\langle g\rangle < \langle a_{t+n+1}\rangle\langle g\rangle < \ldots$$

If $i, j \geqslant t$, then the subgroups $\langle a_i\rangle\langle g\rangle$ and $\langle a_j\rangle\langle g\rangle$, $i \neq j$, cannot be isomorphic, because $|\langle a_i\rangle\langle g\rangle| < |\langle a_j\rangle\langle g\rangle|$. Thus the subgroups in this sequence are pairwise non-isomorphic. Therefore there exists a positive integer m such that $\langle a_m\rangle\langle g\rangle$ is normal in G.

Since D is not a Prüfer group, $A \neq D$. Then $D = A \times B$ (see, for example, [7], Theorem 21.2]). Choose in B a Prüfer q-subgroup Q, then

$$Q = \langle c_n \mid c_1^q = 1, c_{n+1}^q = c_n, n \in \mathbb{N}\rangle$$

(it is possible that $q = p$). Corollary 2.3 shows that every subgroup $\langle c_n\rangle$ is G-invariant. The choice of Q yields that $\langle g\rangle \cap Q = \langle 1\rangle$.

Consider the following sequence of subgroups

$$\langle c_1\rangle\langle g\rangle < \langle c_2\rangle\langle g\rangle < \ldots < \langle c_n\rangle\langle g\rangle < \langle c_{n+1}\rangle\langle g\rangle < \ldots.$$

The subgroups $\langle c_i\rangle\langle g\rangle$ and $\langle c_j\rangle\langle g\rangle$, $i \neq j$, cannot be isomorphic, because $|\langle c_i\rangle\langle g\rangle| < |\langle c_j\rangle\langle g\rangle|$. Thus the subgroups in this sequence are pairwise non-isomorphic. Therefore there exists a positive integer k such that the subgroup $\langle c_k\rangle\langle g\rangle$ is normal in G. It follows that the intersection $\langle g\rangle = \langle a_m\rangle\langle g\rangle \cap \langle c_k\rangle\langle g\rangle$ is normal in G.

Suppose now that the element g has infinite order. Therefore $\langle g\rangle \cap D = \langle 1\rangle$. Since D is not a Prüfer group, we can choose in D two Prüfer subgroups A, C such that $\langle A, C\rangle = A \times C$. Put again

$$A = \langle a_n \mid a_1^p = 1, a_{n+1}^p = a_n, n \in \mathbb{N}\rangle$$

and

$$C = \langle c_n \mid c_1^q = 1, c_{n+1}^q = c_n, n \in \mathbb{N}\rangle$$

(it is possible that $q = p$). Lemma 2.5 shows that, for every $n \in \mathbb{N}$, the subgroups $\langle a_n\rangle$ and $\langle c_n\rangle$ are G-invariant. Consider the following sequences of subgroups

$$\langle a_1\rangle\langle g\rangle < \langle a_2\rangle\langle g\rangle < \ldots < \langle a_n\rangle\langle g\rangle < \langle a_{n+1}\rangle\langle g\rangle < \ldots$$

and

$$\langle c_1\rangle\langle g\rangle < \langle c_2\rangle\langle g\rangle < \ldots < \langle c_n\rangle\langle g\rangle < \langle c_{n+1}\rangle\langle g\rangle < \ldots$$

The subgroups $\langle a_n\rangle\langle g\rangle$ and $\langle a_{n+1}\rangle\langle g\rangle$ (respectively $\langle c_n\rangle\langle g\rangle$ and $\langle c_{n+1}\rangle\langle g\rangle$) cannot be isomorphic, because $|\langle a_n\rangle| < |\langle a_{n+1}\rangle|$ (respectively $|\langle c_n\rangle| < |\langle c_{n+1}\rangle|$). Thus the subgroups in both of these sequences are pairwise non-isomorphic. Therefore there are numbers s, r such that the subgroups $\langle a_s\rangle\langle g\rangle$ and $\langle c_r\rangle\langle g\rangle$ are normal in G. It follows that their intersection $\langle g\rangle = \langle a_s\rangle\langle g\rangle \cap \langle c_r\rangle\langle g\rangle$ is normal in G. □

Now we can say more on the structure of $G \in \mathcal{C}$, if G contains an infinite Chernikov subgroup.

Proposition 2.6 *Suppose that* $G \in \mathcal{C}$ *and that* G *contains an infinite Chernikov subgroup* C. *Then* G *is an* FC*-group. If* G *is a not Dedekind group, then* G *is nilpotent, the set* T *of all elements having finite order is a characteristic Chernikov subgroup, containing* $[G, G]$. *Moreover, the divisible part* D *of* T *is a Prüfer group,* $D \leqslant \zeta(G)$ *and* G/D *is a Dedekind group.*

PROOF — Denote by V the divisible part of C. If V is a not Prüfer group, then Lemma 2.5 shows that G is a Dedekind group. In particular, $[G, G]$ is finite and trivially G is an FC-group. Suppose that V is a Prüfer p-subgroup for some prime p. Then

$$V = \langle d_n \mid d_1^p = 1, d_{n+1}^p = d_n, n \in \mathbb{N} \rangle.$$

By Lemma 2.5 every subgroup of V is G-invariant. In particular, V is normal in G. Let g be an arbitrary element of G. Assume first that g has finite order. Then the subgroup $V\langle g \rangle$ is infinite and locally finite. Using Lemma 2.2 we obtain that $\langle g \rangle V \leqslant FC(G)$, in particular, $g \in FC(G)$. Suppose now that g has infinite order. Then $\langle g \rangle \cap V = \langle 1 \rangle$. Consider the following sequence of subgroups

$$\langle d_1 \rangle \langle g \rangle < \langle d_2 \rangle \langle g \rangle < \ldots < \langle d_n \rangle \langle g \rangle < \langle d_{n+1} \rangle \langle g \rangle < \ldots$$

Again we can see that the subgroups of this sequence are pairwise non-isomorphic. Therefore there exists a positive integer k such that the subgroup $K = \langle d_k \rangle \langle g \rangle$ is normal in G. The subgroup $\langle d_k \rangle$ is finite and G-invariant, and the factor-group $K/\langle d_k \rangle$ is infinite cyclic. It follows that

$$g^x \in g\langle d_k \rangle \quad \text{or} \quad g^x \in g^{-1}\langle d_k \rangle$$

for every element $x \in G$. Since $\langle d_k \rangle$ is finite, it follows that g^G is finite, that is $g \in FC(G)$. Therefore G is an FC-group. Denote by T the set of all elements having finite order. Then T is a (characteristic) subgroup of G, including $[G, G]$ (see, for example, [5], Corollaries 1.5.3 and 1.5.10). Moreover, if D is the divisible part of T, then $D \leqslant \zeta(G)$ (see, for example, [5], Lemma 3.2.9). If we suppose that T is not Chernikov, then Proposition 2.4 shows that G must be a Dedekind group, and we obtain a contradiction. If we suppose that the divisible part of T is not a Prüfer group, then Lemma 2.5 shows that G must be a Dedekind group, and we again obtain a contradiction. Thus the divisible part of T is a Prüfer group.

Let g be an arbitrary element of G. The inclusion $D \leqslant \zeta(G)$ implies that the subgroup $\langle g, D \rangle$ is abelian. Let

$$D = \langle d_n \mid d_1^p = 1, d_{n+1}^p = d_n, n \in \mathbb{N} \rangle.$$

Consider again the sequence of subgroups

$$\langle d_1 \rangle \langle g \rangle < \langle d_2 \rangle \langle g \rangle < \ldots < \langle d_n \rangle \langle g \rangle < \langle d_{n+1} \rangle \langle g \rangle < \ldots.$$

As above we can see that the subgroups of this sequence are pairwise non-isomorphic. Therefore there exists a numbers k such that the subgroup $K = \langle d_k \rangle \langle g \rangle$ is normal in G. It follows that $\langle g \rangle^G$ is abelian. Then G is locally nilpotent (see, for example, [23]). We know that $D \leqslant \zeta(G)$. Since T/D is finite, there is a positive integer m such that $T \leqslant \zeta_m(G)$. And finally, G/T is abelian, so that $G = \zeta_{m+1}(G)$.

Now we show that G/D is a Dedekind group. Let g be an arbitrary element of G. Suppose first that g has finite order. Then the intersection $\langle g \rangle \cap D$ is finite, so that $\langle g \rangle \cap D = \langle d_t \rangle$ for some positive integer t. By Lemma 2.5 every subgroup $\langle d_n \rangle$ is G-invariant. Consider the following sequence of subgroups

$$\langle g \rangle = \langle d_t \rangle \langle g \rangle < \langle d_{t+1} \rangle \langle g \rangle < \ldots < \langle d_{t+n} \rangle \langle g \rangle < \langle d_{t+n+1} \rangle \langle g \rangle < \ldots$$

As we have seen above the subgroups in this sequence are pairwise non-isomorphic. Therefore there exists a number m such that $\langle d_m \rangle \langle g \rangle$ is normal in G. It follows that $D\langle g \rangle$ is normal in G.

If g has infinite order, then $\langle g \rangle \cap D = \langle 1 \rangle$. In this case we consider the following sequence of subgroups

$$\langle d_1 \rangle \langle g \rangle < \langle d_2 \rangle \langle g \rangle < \ldots < \langle d_n \rangle \langle g \rangle < \langle d_{n+1} \rangle \langle g \rangle < \ldots$$

Again the subgroups in this sequence are pairwise non-isomorphic. Therefore there exists a number r such that $\langle d_r \rangle \langle g \rangle$ is normal in G. It follows that $D\langle g \rangle$ is normal in G. Thus every cyclic subgroup of the factor-group G/D is normal in G/D. It follows that G/D is a Dedekind group. $\square$

We are now able to prove Theorem A.

Theorem A *Let G be an infinite locally finite group which is not a Dedekind group. Then $G \in \mathfrak{C}$ if and only if $G = P \times A$ where A is a finite Dedekind group, P is a Sylow p-subgroup of G (p a prime), $\zeta(P)$ contains a Prüfer subgroup D such that P/D is a finite abelian group.*

Proof — Suppose $G \in \mathfrak{C}$ and that G is not a Dedekind group. Then by Proposition 2.6 G is nilpotent and $\zeta(G)$ contains a Prüfer p-subgroup D such that G/D is a finite Dedekind group. Then $G = P \times A$ where P is a p-group and A is a p'-group. Moreover $D \subseteq P$, and A and P/D are finite Dedekind groups. Now we show that P/D is abelian. Suppose that there exist $x, y \in P$ such that $[x, y] \notin D$. Write

$$p^s = max\{o(x), o([x, y])\}.$$

For every $i \in \mathbb{N}$, let $t_i \in D$ be such that

$$o(t_i) = p^{\alpha_i} \geqslant p^{2s},$$

with $\alpha_i < \alpha_j$ if $i < j$. Consider the subgroups $H_i = \langle t_i x \rangle$. We have

$$(t_i x)^{p^s} = t_i^{p^s}$$

of order $\geqslant p^s$, thus $[t_i x, y] = [x, y] \notin \langle t_i x \rangle$, since every subgroup of $\langle t_i x \rangle$ of order $\leqslant p^s$ is contained in $\langle t_i^{p^s} \rangle \subseteq D$. Therefore H_i is not normal in G for every i. Moreover, $|H_i| = o(t_i)$, for every i, hence

$$H_i \not\simeq H_j$$

if $i \neq j$. This contradiction shows that P/D is abelian.

Conversely, assume that G has the required structure. Then D is contained in $\zeta(P)$ and P/D finite implies that P' is finite. Write

$$|P/D| = n \quad \text{and} \quad |P'| = m.$$

If S is a subgroup of G, then

$$S = P_1 \times (A \cap S),$$

where $P_1 \subseteq P$. We have $A \cap S$ normal in G. We show that if P_1 is not normal in G, then $|P_1| \leqslant mn$. In fact, we have $|P_1 D/D| \leqslant n$. Moreover P' is not contained in $P_1 \cap D$ since P_1 is not normal in P. Then $P_1 \cap D < P'$, thus $|P_1 \cap D| < m$. Hence $|P_1| \leqslant mn$, as required. Therefore if S is a non-normal subgroup of G, then the order of S is bounded. Since there exist only finitely many non-isomorphic groups of fixed order, there exist only finitely many non-isomorphic non-normal subgroups of G. The theorem is proved. $\square$

3 The case G non-periodic

In this section we study the structure of a non-periodic group in which the family of all non-normal subgroups has finite isomorphic type. We start with two very useful results.

Lemma 3.1 *Suppose that* $G \in \mathfrak{C}$. *If* G *contains a free abelian subgroup of infinite* 0*-rank, then* G *is an abelian group.*

Proof — Let A be a free abelian subgroup of G with infinite 0-rank. Without loss of generality we may suppose that $r_0(A)$ is countable. Then $A = \mathrm{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$, where a_n is an element of infinite order for all $n \in \mathbb{N}$. Let v be an arbitrary element of A, then there is a positive integer m such that $\langle v \rangle \cap \mathrm{Dr}_{n \geqslant m} \langle a_n \rangle = \langle 1 \rangle$. Then the following subgroups

$$\langle v \rangle \times \langle a_m \rangle, \langle v \rangle \times \langle a_m \rangle \times \langle a_{m+2} \rangle, \dots,$$
$$\langle v \rangle \times \langle a_m \rangle \times \langle a_{m+2} \rangle \times \dots \times \langle a_{m+2n} \rangle, n \in \mathbb{N},$$

are pairwise non-isomorphic. It follows that there is a positive integer k such that the subgroup

$$\langle v \rangle \times \langle a_m \rangle \times \langle a_{m+2} \rangle \times \dots \times \langle a_{m+2k} \rangle$$

is normal in G. Using the same arguments, we obtain that there is a positive integer t such that the subgroup

$$\langle v \rangle \times \langle a_{m+1} \rangle \times \langle a_{m+3} \rangle \times \dots \times \langle a_{m+2t+1} \rangle$$

is normal in G. Then from the obvious equality

$$\big(\langle v \rangle \times \langle a_m \rangle \times \dots \times \langle a_{m+2k} \rangle\big) \cap \big(\langle v \rangle \times \langle a_{m+1} \rangle \times \dots \times \langle a_{m+2t+1} \rangle\big) = \langle v \rangle$$

we obtain that the subgroup $\langle v \rangle$ is normal in G. In particular, the subgroup $\langle a_n \rangle$ is G-invariant for every $n \in \mathbb{N}$. Let g be an arbitrary element of G. Then there is a positive integer k such that

$$\langle g \rangle \cap \mathop{\mathrm{Dr}}_{n \geqslant k} \langle a_n \rangle = \langle 1 \rangle.$$

The fact that each subgroup $\langle a_n \rangle$ is G-invariant implies that

$$\left\langle g, \mathop{\mathrm{Dr}}_{k \leqslant n \leqslant k+j} \langle a_n \rangle \right\rangle = \left(\mathop{\mathrm{Dr}}_{k \leqslant n \leqslant k+j} \langle a_n \rangle \right) \rtimes \langle g \rangle,$$

for any $j \in \mathbb{N}$. Repeating now the above arguments, we obtain that the subgroup $\langle g \rangle$ is normal in G. The fact that each cyclic subgroup of G is normal implies that every subgroup of G is normal in G. Being non-periodic, G is abelian (see for example [1]). $\square$

Lemma 3.2 *Suppose that* $G \in \mathfrak{C}$. *If* L *is a torsion-free nilpotent subgroup of* G*, then* L *is abelian.*

PROOF — Suppose the contrary and let L be non-abelian. Then

$$\zeta_2(L) \neq \zeta(L).$$

Choose an element $a \in \zeta_2(L) \setminus \zeta(L)$. Then there exists an element b such that $[a, b] = c \neq 1$. Let p be a prime and $r = p^2$. Put

$$H_p = \langle a^r, b, c^p \rangle,$$

then clearly $[H_p, H_p] = \langle c^r \rangle$, $\zeta(H_p) = \langle c^p \rangle$, so that $\zeta(H_p)/[H_p, H_p]$ is a group of order p. It follows that if q is a prime, $q \neq p$, then the subgroups H_p and H_q cannot be isomorphic. Choose an infinite set π of primes. Then in the family $\{H_p | p \in \pi\}$ every two subgroups are not pairwise isomorphic. Since $[a, b] = c \notin H_p$, H_p cannot be normal in G for every prime p, and we obtain a contradiction. This contradiction shows that L is abelian. □

We can now describe the structure of a non-periodic group G in $\mathfrak{C}$ if G contains a locally finite infinite subgroup.

Theorem 3.3 *Suppose that* G *is a non-periodic group in* $\mathfrak{C}$*, and that* G *contains an infinite locally finite subgroup. Then either* G *is an abelian group or* $\zeta(G)$ *contains a Prüfer subgroup* D *such that* G/D *is an abelian minimax group having finite* 0*-rank and finite periodic part. Moreover,* G *is central-by-finite.*

PROOF — Suppose that is not abelian, then G is not a Dedekind group, since G is non-periodic (see for example [1]). Then Proposition 2.4 shows that every locally finite subgroup of G must be Chernikov. Moreover Proposition 2.6 shows that $\zeta(G)$ contains a Prüfer subgroup D such that G/D is a Dedekind group with finite periodic part. Being non-periodic, G/D is abelian (see for example [1]). Suppose that $r_0(G/D)$ is infinite. Then G/D contains a free abelian subgroup A/D of infinite countable 0-rank. In this case A contains a free abelian subgroup B of infinite countable 0-rank (see, for example, [10]). But then Lemma 3.1 implies that G is an abelian group. This contradiction shows that $r_0(G/D)$ is finite. By Proposition 2.6 G is nilpotent and an FC-group. Then $G/\zeta(G)$ is periodic ([24], Theorem 1.4). Since $\zeta(G)$ has finite 0-rank, it contains a finitely generated torsion-free subgroup B such that $\zeta(G)/B$ is periodic. Then G/B is also periodic. Put $C = B^8$. The factor group G/C is periodic, moreover its Sylow p-subgroups are Chernikov for each prime p. Suppose that G is not minimax. Then the set $\Pi(G/C)$ is infinite. Since the subgroup $T = \mathrm{Tor}(G)$ is Chernikov, the subset $\Pi(G/C) \setminus \Pi(T)$ is infinite. Let g be an arbitrary element of G. Since the subgroup $\langle gC\rangle$ is finite, we can choose a subset π of $\Pi(G/C)$ such that $\Pi(G/C) \setminus \pi$ is finite and

$$\emptyset = \pi \cap (\Pi(T) \cup \Pi(\langle gC\rangle)).$$

Choose in π two infinite subsets π_1 and π_2 such that

$$\pi_1 \cup \pi_2 = \pi \quad \text{and} \quad \pi_1 \cap \pi_2 = \emptyset.$$

Since the subset π_1 is infinite, it is possible to choose in π_1 a family

$$\{\rho_n \mid n \in \mathbb{N}\}$$

of infinite subsets ρ_n such that $\bigcup_{n\in\mathbb{N}} \rho_n = \pi_1$ and $\rho_n \cap \rho_m = \emptyset$ whenever $n \neq m$. Consider the ascending chain

$$S_0 = C \leqslant S_1 \leqslant \ldots \leqslant S_n \leqslant S_{n+1} \leqslant \ldots$$

of subgroups, defined by the rule: S_1/C is the Sylow ρ_1-subgroup of G/C, S_2/C is the Sylow $(\rho_1 \cup \rho_2)$-subgroup of G/C, S_n/C is the Sylow $(\rho_1 \cup \ldots \cup \rho_n)$-subgroup of G/C, $n \in \mathbb{N}$. Since G is nilpotent and $S_n \cap T = \langle 1\rangle$, S_n is a normal abelian torsion-free subgroup of G, for all $n \in \mathbb{N}$. The choice of S_n yields that

$$\langle g, S_n\rangle/C = \langle gC, S_n/C\rangle = \langle gC\rangle \times S_n/C,$$

in particular, $\langle gC, S_n/C\rangle$ is abelian, $n \in \mathbb{N}$. The fact that $[G, G]$ is periodic implies that $\langle g, S_n\rangle$ is abelian. Suppose that the subgroups $\langle g, S_n\rangle$ and $\langle g, S_{n+k}\rangle$, $k \geqslant 1$, are isomorphic. Let

$$f : \langle g, S_n\rangle \to \langle g, S_{n+k}\rangle$$

be an isomorphism. Let $E = f(C)$, then $\langle g, S_{n+k}\rangle/E \simeq \langle g, S_n\rangle/C$. It follows that

$$\Pi\big(\langle g, S_{n+k}\rangle/E\big) = \Pi\big(\langle g, S_n\rangle/C\big) = \Pi(S_n/C) \cup \Pi\big(\langle gC\rangle\big).$$

Since $r_0(E) = r_0(C)$, then both factors $E/(E \cap C)$ and $C/(E \cap C)$ are finite. Then

$$\Pi\big(\langle g, S_{n+k}\rangle/(E \cap C)\big) = \Pi(S_n/C) \cup \Pi\big(\langle gC\rangle\big) \cup \Pi\big(E/(E \cap C)\big),$$

where the last subset is finite. Since the subset $\Pi(S_{n+k}/C) \setminus \Pi(S_n/C)$ is infinite, we obtain a contradiction. This contradiction shows that the subgroups $\langle g, S_n\rangle$ and $\langle g, S_{n+k}\rangle$ cannot be isomorphic. Then there exists a number t such that the subgroups $\langle g, S_n\rangle$ are normal in G for all $n \geqslant t$. Using the same arguments, we can choose a subgroup U of G such that $\Pi(U/C) \subseteq \pi_2$ and $\langle g, U\rangle$ is normal in G. By such a choice $\langle g, S_n\rangle/C \cap \langle g, U\rangle/C = \langle gC\rangle$, which yields that $\langle gC\rangle$ is normal in G/C. In other words, every cyclic subgroup of G/C is normal in G/C. Therefore every subgroup of G/C is normal in G/C. Since the factor-group G/C contains an element of order 8, G/C cannot be a non-abelian Dedekind group (see [1]). Thus G/C is abelian. Let p be a prime, put $C_1 = C^p, C_{n+1} = C_n^p$, $n \in \mathbb{N}$. By such a choice

$$\bigcap_{n\in\mathbb{N}} C_n = \langle 1\rangle.$$

Repeating the above arguments, we obtain that G/C_n is abelian for all $n \in \mathbb{N}$. In other words, $[G, G] \leqslant C_n$, for all $n \in \mathbb{N}$. Then

$$[G, G] \leqslant \bigcap_{n\in\mathbb{N}} C_n = \langle 1\rangle,$$

from which it follows that G is abelian, the final contradiction.

Now we show that G is central-by-finite. We know that $G' \leqslant D$, a Prüfer group, say a Prüfer p-group, where p is a prime. We will show that there exists a positive integer n such that $[x, y]^{p^n} = 1$, for every $x, y \in G$. From this it will follow that $G/\zeta(G)$ is a periodic minimax group of finite exponent and then it is finite, as required. Since

$$G = \langle a \mid a \text{ is torsion-free}\rangle,$$

it is enough to show that there exists a positive integer n such that

$$[x, y]^{p^n} = 1,$$

for any torsion-free elements $x, y \in G$. Assume not, then there exist positive integers

$$n_1 < n_2 < \ldots < n_s < \ldots$$

and torsion-free elements x_i, y_i of G such that $[x_i, y_i]^{p^{n_i}} \neq 1$. Let $c_i \in D$, $o(c_i) = p^{n_i}$, and consider the subgroups

$$T_i = \langle x_i, c_i \rangle = \langle x_i \rangle \times \langle c_i \rangle.$$

Then $\langle c_i \rangle = T_i'$, thus T_i and T_j are not isomorphic if $i \neq j$. Moreover, for every i, T_i is not normal in G, because $[x_i, y_i] \notin T_i$, since $T_i \cap G' = \langle c_i \rangle$ and $c_i^{p^{n_i}} = 1$ while $[x_i, y_i]^{p^{n_i}} \neq 1$, and we have a contradiction $\square$

From now on we assume that G is a generalized radical group. By Proposition 2.4 and Theorem 3.3 we can suppose that every periodic subgroup of G is finite.

Lemma 3.4 *Suppose that G is a non-periodic locally generalized radical group in $\mathfrak{C}$, and that all periodic subgroups of G are finite. If G is non-abelian, then G is a soluble-by-finite group of finite 0-rank.*

Proof — First assume that G is a generalized radical group. Let D be the maximal normal radical subgroup of G. Since G is non-abelian, by Lemma 3.1 every abelian subgroup of D has finite 0-rank. Moreover the torsion subgroup of every abelian subgroup is finite. By a Theorem by Charin (see [3], Theorem 8) D is a soluble group of finite 0-rank, in particular D is a radical group of finite 0-rank. Let L/D be the maximal normal locally finite subgroup of G/D. Then L has finite 0-rank. Using Theorem 2.4.13 of [5] (see also [19]) we obtain that L is soluble-by-finite. In particular, it follows that L/D is finite. In turn out that G/D is finite ([5], Lemma 3.4.1). Now suppose G locally generalized radical. If every finitely generated subgroup of G is normal in G, then clearly G is a Dedekind group and being non-periodic, G is abelian (see [1]). Therefore suppose that G includes a finitely generated subgroup D_1 which is not normal in G. Then D_1 is generalized radical thus D_1 is a soluble-by-finite group of finite 0-rank. Using the same arguments, we obtain that every finitely generated subgroup F of G including D_1 is a soluble-by-finite group having finite 0-rank. If $r_0(F) = r_0(D_1)$ for each finitely generated subgroup F of G including D_1, then $G/\mathrm{Tor}(G)$ is a soluble-by-finite group of finite 0-rank. Our assumption concerning all periodic subgroups of G implies that $\mathrm{Tor}(G)$ is finite, so that G also is a soluble-by-finite group of finite 0-rank. Therefore assume that G contains a finitely generated subgroup D_2 including D_1 such that $r_0(D_2) > r_0(D_1)$. If we suppose that $r_0(F) = r_0(D_2)$ for each finitely generated subgroup F

of G including D_2, then repeating the above arguments we again obtain that G is a soluble-by-finite group of finite 0-rank. Thus we can assume that there exists an infinite chain of finitely generated subgroups

$$D_1 \leqslant D_2 \leqslant \ldots \leqslant D_n \leqslant D_{n+1} \leqslant \ldots$$

such that $r_0(D_{n+1}) > r_0(D_n)$ for each $n \in \mathbb{N}$. By such a choice the subgroups D_n and D_{n+1} cannot be isomorphic for each $n \in \mathbb{N}$. It follows that there exists a positive integer t such that the subgroups D_n are normal in G for each $n \geqslant t$. Put $D = \bigcup_{n \in \mathbb{N}} D_n$. Then D has an ascending series of normal subgroups, whose factors are soluble-by-finite. It follows that D is a generalized radical group. Let A be an arbitrary abelian subgroup of D. If we assume that $r_0(A)$ is finite, then since $\mathrm{Tor}(A)$ is finite we obtain that D is a soluble-by-finite group, having finite 0-rank, and we obtain a contradiction. This contradiction shows that D contains an abelian subgroup B of infinite 0-rank. Then $B/\mathrm{Tor}(B)$ contains a free abelian subgroup $C/\mathrm{Tor}(B)$ of infinite 0-rank. Since $\mathrm{Tor}(B)$ is finite, $C = \mathrm{Tor}(B) \times E$ (see for example [7], Theorem 27.5), where the subgroup E is a free abelian subgroup of infinite 0-rank, and an application of Lemma 3.1 shows that G is abelian. This contradiction proves the result. □

Let G be a group and A be a normal abelian Chernikov subgroup of G. We say that A is G-*quasifinite* if every proper G-invariant subgroup of A is finite. The following lemma is very well-known.

Lemma 3.5 *Let* G *be a nilpotent group and* A *be a normal abelian Chernikov subgroup of* G. *If* A *is* G*-quasifinite, then* $A \leqslant \zeta(G)$.

PROOF — Obviously A is divisible. Suppose that $C_G(A) \neq G$. Choose an element g such that

$$C_G(A) \neq gC_G(A) \in \zeta(G/C_G(A)).$$

Then it is not difficult to prove that the subgroups $[A, g]$ and $C_A(g)$ are G-invariant. Moreover the mapping $a \longmapsto [a, g]$, $a \in A$, is an endomorphism of A, so that $[A, g] \simeq A/C_A(g)$. Since G is nilpotent, we have $A \neq [A, g]$, which implies that $[A, g]$ is finite. We have above noted that A is divisible. Then A does not contain proper subgroups of finite index. Hence the finiteness of $A/C_A(g)$ means that $A = C_A(g)$, and we obtain a contradiction with the choice of g. This contradiction proves that $A \leqslant \zeta(G)$. □

Let A be an abelian group having finite Prüfer rank. Choose in A a finitely generated subgroup B such that $r_0(A) = r_0(B)$, then A/B is periodic. Recall (see for example [22]) that the set $Sp(A)$ of all primes p such that the Sylow p-subgroup of A/B is infinite is an invariant of A, called the *spectrum* of the group A.

The following two lemmas are very important in our investigation.

Lemma 3.6 *Let* $G_1 = A_1 \rtimes \langle g_1 \rangle$ *and* $G_2 = A_2 \rtimes \langle g_2 \rangle$ *be groups such that* A_1 *and* A_2 *are torsion-free abelian groups of finite* 0*-rank* r, g_1 *and* g_2 *are elements of infinite orders. Suppose also that* A_1 *(respectively* A_2*) contains a* G_1*-invariant (respectively* G_2*-invariant) subgroup* D_1 *(respectively* D_2*) such that* A_1/D_1 *(respectively* A_2/D_2*) is a periodic divisible abelian group. If* $\Pi(A_1/D_1) \neq \Pi(A_2/D_2)$, *then the groups* G_1 *and* G_2 *cannot be isomorphic.*

Proof — Suppose the contrary, then there is an isomorphism

$$f : G_1 \to G_2.$$

By our conditions the locally nilpotent radical $L\mathcal{N}(G_1)$ of G_1 (respectively G_2) contains A_1 (respectively A_2).

Suppose first that $L\mathcal{N}(G_1) = A_1$, then the factor group $G_1/L\mathcal{N}(G_1)$ is infinite. Since $f(L\mathcal{N}(G_1)) = L\mathcal{N}((G_2)$, the factor group $G_2/L\mathcal{N}((G_2)$ also must be infinite. The inclusion $A_2 \leqslant L\mathcal{N}(G_2)$ implies that

$$A_2 = L\mathcal{N}(G_2).$$

The subgroup $D_3 = f(D_1)$ must be G_2-invariant and the factor A_2/D_3 must be periodic and divisible. In particular, it follows that

$$r_0(D_3) = r_0(A_2) = r_0(D_2).$$

Since the subgroups D_2, D_3 are finitely generated, it follows that both the factors $D_2/(D_3 \cap D_2)$ and $D_3/(D_2 \cap D_3)$ are finite. Since the factor group A_2/D_2 is divisible, we obtain that

$$A_2/(D_3 \cap D_2) = F_1/(D_3 \cap D_2) \times P_1/(D_3 \cap D_2)$$

where $F_1/(D_3 \cap D_2)$ is finite, $P_1/(D_3 \cap D_2)$ is divisible and

$$\Pi\big(P_1/(D_3 \cap D_2)\big) = \Pi(A_2/D_2).$$

On the other hand, the fact that the group A_2/D_3 is divisible and

$D_3/(D_2 \cap D_3)$ is finite implies that

$$A_2/(D_3 \cap D_2) = F_2/(D_3 \cap D_2) \times P_2/(D_3 \cap D_2)$$

where $F_2/(D_3 \cap D_2)$ is finite, $P_2/(D_3 \cap D_2)$ is divisible and

$$\Pi(P_2/(D_3 \cap D_2)) = \Pi(A_2/D_3).$$

We note that the largest divisible subgroup of a periodic abelian group is unique. It follows that $P_1/(D_3 \cap D_2) = P_2/(D_3 \cap D_2)$. Since f is an isomorphism, $\Pi(A_2/D_3) = \Pi(A_1/D_1)$. Thus we have

$$\Pi(A_1/D_1) = \Pi\big(P_1/(D_3 \cap D_2)\big) = \Pi(A_2/D_2),$$

and we obtain a contradiction with $\Pi(A_1/D_1) \neq \Pi(A_2/D_2)$.

Suppose now that that

$$L\mathcal{N}(G_1) \neq A_1.$$

Then

$$L\mathcal{N}(G_1) = L_1 = A_1 \rtimes \langle x_1 \rangle$$

where $\langle x_1 \rangle = L\mathcal{N}(G_1) \cap \langle g_1 \rangle$. Since L_1 is a torsion-free subgroup of finite 0-rank, it is nilpotent [6, Corollary 2.3.4]. We have

$$L_1/D_1 = (A_1/D_1)\langle x_1 D_1 \rangle$$

where $A_1/D_1 = \mathrm{Dr}_{p\in\tau} S_p/D_1$ where $\tau = \Pi(A_1/D_1)$ and S_p/D_1 is the Sylow p-subgroup of A_1/D_1, $p \in \Pi(A_1/D_1)$. We note that S_p/D_1 is a divisible Chernikov p-subgroup, $p \in \Pi(A_1/D_1)$. Being Chernikov, S_p/D_1 satisfies the minimal condition on all subgroups. It follows that there exists a finite series

$$D_1 = R_{(p,1)} \leqslant R_{(p,2)} \leqslant \ldots \leqslant R_{(p,s)} = S_p$$

of $\langle x_1 \rangle$-invariant subgroups, whose factors

$$R_{(p,2)}/R_{(p,1)}, \ldots, R_{(p,s)}/R_{(p,s-1)}$$

are $\langle x_1 \rangle$-quasifinite. By Lemma 3.5 all these factors are $\langle x_1 \rangle$-central. In particular,

$$[S_p, x_1] \leqslant R_{(p,s-1)}.$$

It follows that the factor $\langle S_p, x_1\rangle/R_{(p,s-1)}$ is abelian. Since this is true for each $p \in \Pi(A_1/D_1)$,

$$\langle A_1, x_1\rangle / \underset{p\in\tau}{\mathrm{Dr}}\, R_{(p,s-1)}$$

is abelian. It follows that

$$Sp\big(L_1/[L_1, L_1]\big) = \Pi(A_1/D_1).$$

We note that $L_2 = f(L\mathcal{N}(G_1))$ is the locally nilpotent radical of G_2. Since G_1/L_1 is finite, G_2/L_2 likewise is finite. Then $L\mathcal{N}(G_2) \neq A_2$, so we obtain that

$$L_2 = A_2 \rtimes \langle x_2\rangle$$

where $\langle x_2\rangle = L_2 \cap \langle g_2\rangle$. Using the above arguments, we obtain that

$$Sp\big(L_2/[L_2, L_2]\big) = Sp(A_2/D_2).$$

The isomorphism $L_1 \simeq L_2$ implies that the factor groups $L_1/[L_1, L_1]$ and $L_2/[L_2, L_2]$ must be isomorphic. It follows that

$$\Pi(A_1/D_1) = Sp\big(L_1/[L_1, L_1]\big) = Sp\big(L_2/[L_2, L_2]\big) = \Pi(A_2/D_2),$$

and we obtain a contradiction. This contradiction proves the result. □

Lemma 3.7 *Let* $G_1 = A_1 \rtimes \langle g_1\rangle$ *and* $G_2 = A_2 \rtimes \langle g_2\rangle$ *be groups such that* A_1 *and* A_2 *are torsion-free abelian groups of finite* 0*-rank* r, g_1 *and* g_2 *are elements of infinite orders. Suppose also that* A_1 *(respectively* A_2*) contains a* G_1*-invariant (respectively* G_2*-invariant) subgroup* D_1 *(respectively* D_2*) such that* A_1/D_1 *(respectively* A_2/D_2*) is a periodic abelian group with finite Sylow* p*-subgroups for all primes* p*. If* $\Pi(A_1/D_1)$ *and* $\Pi(A_2/D_2) \setminus \Pi(A_1/D_1)$ *are infinite, then the groups* G_1 *and* G_2 *cannot be isomorphic.*

Proof — Suppose the contrary, then there is an isomorphism

$$f : G_1 \to G_2.$$

By our conditions the locally nilpotent radical $L\mathcal{N}(G_1)$ of G_1 (respectively G_2) contains A_1 (respectively A_2).

Suppose first that $L\mathcal{N}(G_1) = A_1$, then the factor group $G_1/\mathcal{N}(G_1)$ is infinite. Since $f(L\mathcal{N}(G_1)) = L\mathcal{N}(G_2)$, the factor group $G_2/L\mathcal{N}(G_2)$

also must be infinite. The inclusion $A_2 \leqslant L\mathcal{N}(G_2)$ implies that

$$A_2 = L\mathcal{N}(G_2).$$

The subgroup $D_3 = f(D_1)$ has to be G_2-invariant and the factor group A_2/D_3 periodic. In particular, it follows that

$$r_0(D_3) = r_0(A_2) = r_0(D_2).$$

Since the subgroups D_2, D_3 are finitely generated, it follows that both factors $D_2/(D_3 \cap D_2)$ and $D_3/(D_2 \cap D_3)$ are finite. Then

$$\Pi\big(A_2/(D_3 \cap D_2)\big) = \Pi(A_2/D_2) \cup \pi_1$$

for some finite disjoint set π_1 of primes. In a similar way, the same set $\Pi(A_2/(D_3 \cap D_2)) = \Pi(A_2/D_3) \cup \pi_2$ for some finite disjoint set π_2 of primes. Since f is an isomorphism, $\Pi(A_2/D_3) = \Pi(A_1/D_1)$. Then we have

$$\Pi(A_2/D_2) \setminus \Pi(A_1/D_1) = \Pi(A_2/D_2) \setminus \Pi(A_2/D_3) =$$
$$\big(\Pi(A_2/(D_3 \cap D_2)) \setminus \pi_1\big) \setminus \big(\Pi(A_2/(D_3 \cap D_2)) \setminus \pi_2\big).$$

Since the last set is finite, we obtain a contradiction.

Suppose now that $L\mathcal{N}(G_1) \neq A_1$. Then

$$L\mathcal{N}(G_1) = L_1 = A_1 \rtimes \langle x_1 \rangle$$

where $\langle x_1 \rangle = L\mathcal{N}(G_1) \cap \langle g_1 \rangle$. Since L_1 is a torsion-free subgroup having finite 0-rank, then it is nilpotent (see [5], Corollary 2.3.4). We have

$$L_1/D_1 = (A_1/D_1)\langle x_1 D_1 \rangle \quad \text{and} \quad A_1/D_1 = \operatorname*{Dr}_{p \in \tau} S_p/D_1$$

where $\tau = \Pi(A_1/D_1)$ and S_p/D_1 is the finite Sylow p-subgroup of A_1/D_1, $p \in \Pi(A_1/D_1)$. Since $\langle S_p/D_1, x_1 D_1 \rangle$ is nilpotent, then

$$[S_p/D_1, x_1 D_1] = R_p/D_1 \neq S_p/D_1.$$

It follows that the factor $\langle S_p, x_1 \rangle / R_p$ is abelian and its periodic part is a finite p-subgroup. Since that is true for each $p \in \Pi(A_1/D_1)$, then $\langle A_1, x_1\rangle/\mathrm{Dr}_{p\in\tau}R_p$ is abelian and

$$\Pi\left(\langle A_1, x_1\rangle/\operatorname*{Dr}_{p\in\tau} R_p\right) = \Pi(A_1/D_1).$$

The fact that $L_1/\mathrm{Dr}_{p\in\tau}R_p$ is abelian implies the inclusion

$$[L_1, L_1] \leqslant \operatorname*{Dr}_{p\in\tau} R_p.$$

On the other hand, $D_1 \leqslant \mathrm{Dr}_{p\in\tau}R_p$, so that $[L_1, L_1]D_1 \leqslant \mathrm{Dr}_{p\in\tau}R_p$. Hence we obtain that $A_1/[L_1, L_1]D_1$ is periodic and

$$\Pi\big(A_1/[L_1, L_1]D_1\big) = \Pi(A_1/D_1).$$

Since the factor group L_1/A_1 is infinite cyclic, we obtain the direct decomposition $L_1/[L_1, L_1]D_1 = (A_1/[L_1, L_1]D_1) \times \langle x_1[L_1, L_1]D_1\rangle$. Thus, the factor $L_1/[L_1, L_1]$ includes a finitely generated subgroup

$$\langle x_1, D_1, [L_1, L_1]\rangle/[L_1, L_1] = B_1/[L_1, L_1]$$

such that $(L_1/[L_1, L_1])/(B_1/[L_1, L_1])$ is periodic and

$$\Pi(A_1/D_1) = \Pi\big((L_1/[L_1, L_1])/(B_1/[L_1, L_1])\big).$$

We note that $L_2 = f(L\mathcal{N}(G_1))$ is the locally nilpotent radical of G_2. Since G_1/L_1 is finite, then G_2/L_2 likewise is finite. Then $L\mathcal{N}(G_2) \neq A_2$, so we obtain that $L_2 = A_2 \rtimes \langle x_2\rangle$ where $\langle x_2\rangle = L_2 \cap \langle g_2\rangle$. Repeating the above arguments, we obtain that $L_2/[L_2, L_2]$ includes a finitely generated subgroup $B_2/[L_2, L_2]$ such that $(L_2/[L_2, L_2])/(B_2/[L_2, L_2]))$ is periodic and

$$\Pi\big((L_2/[L_2, L_2])/(B_2/[L_2, L_2])\big) = \Pi(A_2/D_2).$$

The isomorphism $L_1 \simeq L_2$ implies that the factor groups $L_1/[L_1, L_1]$ and $L_2/[L_2, L_2]$ must be isomorphic. Now we can use the above arguments and obtain a contradiction. This contradiction proves the result. $\square$

Now we are able to give a description of non-periodic locally generalized groups G in the class $\mathcal{C}$ if all subgroups of G are finite.

Proposition 3.8 *Let $G \in \mathfrak{C}$ be a non-periodic locally generalized radical group. Suppose that all periodic subgroups of G are finite. If G is non-abelian, then G is a soluble-by-finite minimax group. Therefore, if G is not abelian-by-finite, then G has normal subgroups $A \leqslant K \leqslant G$ where A is an abelian minimax torsion-free subgroup, K/A is an abelian finitely generated torsion-free group and G/K is finite.*

Proof — Lemma 3.4 shows that G is a soluble-by-finite group of finite 0-rank. If every abelian subgroup of G is minimax then G itself is minimax (see [2], [26]). Therefore assume that G contains an abelian subgroup B which is not minimax. By our assumption $\mathrm{Tor}(B)$ is finite, hence

$$B = \mathrm{Tor}(B) \times A$$

(see for example [7], Theorem 27.5) where the subgroup A is torsion-free and not minimax. Let C be a free abelian subgroup of A such that A/C is periodic. Since A has finite 0-rank, the subgroup C is finitely generated. Put $D = C^8$. The factor group A/D is periodic. Since A is not minimax, the set $\Pi(A/D)$ is infinite.

Suppose first that the set $\sigma = Sp(A)$ is infinite. Choose two infinite subsets σ_1 and σ_2 of σ such that $\sigma_1 \cup \sigma_2 = \sigma$ and $\sigma_1 \cap \sigma_2 = \emptyset$. Let

$$\sigma_1 = \{p_n | n \in \mathbb{N}\},$$

and denote by P_k/D the Sylow $\{p_1, \dots, p_k\}$-subgroup of A/D, $k \in \mathbb{N}$. Then

$$Sp(P_k) = \{p_1, \dots, p_k\},$$

and from that we get that the subgroups P_k and P_m are not isomorphic whenever $k \neq m$. Then there exists a number t such that the subgroups P_k are normal in G for all $k \geqslant t$. Using similar arguments, we can find a subgroup Q of A such that $\Pi(Q/D) \subseteq \sigma_2$ and Q is normal in G. By such a choice $D = P_t \cap Q$ is normal in G. Let g be an arbitrary element of G. If the element gD has infinite order, then $\langle gD \rangle \cap A/D = \langle 1 \rangle$. If gD has finite order, then we can choose a subgroup A_1/D of A/D such that $\Pi(A/D) \setminus \Pi(A_1/D)$ is finite and

$$\langle 1 \rangle = \langle gD \rangle \cap A_1/D.$$

In particular, the set $\Pi(A_1/D)$ is infinite. In this case instead of A we can consider the subgroup A_1. Therefore without loss of generality we can assume that $\langle 1 \rangle = \langle gD \rangle \cap A/D$. Since the subgroups P_k are normal in G for all $k \geqslant t$, then

$$\langle gD, P_k/D\rangle = P_k/D \rtimes \langle gD\rangle.$$

If gD has finite order, it is not hard to prove that the subgroups $\langle g, P_k\rangle$ and $\langle g, P_m\rangle$ are not isomorphic whenever $k, m \geqslant t, k \neq m$. If gD has infinite order, then an application of Lemma 3.6 shows that the subgroups $\langle g, P_k\rangle$ and $\langle g, P_m\rangle$ are not isomorphic whenever $k, m \geqslant t, k \neq m$. It follows that there exists a positive integer $t_1 \geqslant t$ such that the subgroups $\langle g, P_k\rangle$ are normal in G for $k \geqslant t_1$. Using the same arguments, we can choose a subgroup Q_1 of A such that

$$\Pi(Q_1/D) \subseteq \sigma_2$$

and $\langle Q_1, g\rangle$ is normal in G. It follows that $\langle gD, P_k/D\rangle$ and $\langle gD, Q_1/D\rangle$ are normal in G/D, $k \geqslant t_1$. The choice of g shows that

$$\langle gD\rangle = \langle gD, P_k/D\rangle \cap \langle gD, Q_1/D\rangle$$

is normal in G/D. In other words, every cyclic subgroup of G/D is normal in G/D. Then every subgroup of G/D is normal in G/D. Since the factor group G/D contains an element of order 8, G/D cannot be a non-abelian Dedekind group (see [1]). Thus G/D is abelian. Let p be a prime, put $D_1 = D^p, D_{n+1} = D_n^p, n \in \mathbb{N}$. By such a choice

$$\bigcap_{n\in\mathbb{N}} D_n = \langle 1\rangle.$$

Repeating the above arguments, we obtain that G/D_n is abelian for all $n \in \mathbb{N}$. In other words, $[G, G] \leqslant D_n$ for all $n \in \mathbb{N}$. Then

$$[G, G] \leqslant \bigcap_{n\in\mathbb{N}} D_n = \langle 1\rangle,$$

therefore G is abelian, a contradiction.

So, suppose now that $Sp(A)$ is finite. In this case the subset

$$\Pi(A/D) \setminus Sp(A) = \pi$$

is infinite. Let V/D be the Sylow π-subgroup of A/D. By its choice the Sylow p-subgroup of V/D is finite for each prime $p \in \Pi(V/D)$. Choose in π two infinite subsets π_1 and π_2 such that $\pi_1 \cup \pi_2 = \pi$

and $\pi_1 \cap \pi_2 = \emptyset$. Since the subset π_1 is infinite, it is possible to choose in π_1 a family $\{\rho_n | n \in \mathbb{N}\}$ of infinite subsets ρ_n such that

$$\bigcup_{n \in \mathbb{N}} \rho_n = \pi_1$$

and $\rho_n \cap \rho_m = \emptyset$ whenever $n \neq m$. Consider the ascending chain

$$V_0 = D \leqslant V_1 \leqslant \ldots \leqslant V_n \leqslant V_{n+1} \leqslant \ldots$$

of subgroups of V, defined by the rule: V_1/D is the Sylow ρ_1-subgroup of V/D, V_2/D is the Sylow $(\rho_1 \cup \rho_2)$-subgroup of V/D, V_n/D is the Sylow $(\rho_1 \cup \ldots \cup \rho_n)$-subgroup of V/D, $n \in \mathbb{N}$. It is not hard to prove that the subgroups V_k and V_m are not isomorphic for $k \neq m$. Then there exists a number t such that the subgroups V_k are normal in G for all $k \geqslant t$. Using the same arguments, we can choose a subgroup U of V such that $\Pi(U/D) \subseteq \pi_2$ and U is normal in G. By this choice $D = V_t \cap U$ is normal in G. Let g be an arbitrary element of G. If the element gD has infinite order, then $\langle gD \rangle \cap V/D = \langle 1 \rangle$. If gD has finite order, then we can choose a subgroup W/D of V/D such that $\Pi(V/D) \setminus \Pi(W/D)$ is finite and $\langle 1 \rangle = \langle gD \rangle \cap W/D$. In particular, the set $\Pi(W/D)$ is infinite. In this case instead of V we can consider the subgroup W. Therefore without loss of generality we can assume that $\langle 1 \rangle = \langle gD \rangle \cap V/D$. Since the subgroups V_k are normal in G for all $k \geqslant t$, then

$$\langle gD, V_k/D \rangle = V_k/D \rtimes \langle gD \rangle.$$

If gD has finite order, it is not hard to prove that the subgroups $\langle g, V_k \rangle$ and $\langle g, V_m \rangle$ are not isomorphic whenever $k, m \geqslant t$, $k \neq m$. If gD has infinite order, then we can apply Lemma 3.7 and obtain that the subgroups $\langle g, V_k \rangle$ and $\langle g, V_m \rangle$ are not isomorphic whenever $k, m \geqslant t$, $k \neq m$. It follows that there exists a number $t_1 \geqslant t$ such that the subgroups $\langle g, V_k \rangle$ are normal in G for $k \geqslant t_1$. Using the same arguments, we can choose a subgroup U_1 of V such that $\Pi(U_1/D) \subseteq \pi_2$ and $\langle U_1, g \rangle$ is normal in G. It follows that $\langle gD, V_k/D \rangle$ and $\langle gD, U_1/D \rangle$ are normal in G/D, $k \geqslant t_1$. The choice of g shows that

$$\langle gD \rangle = \langle gD, V_k/D \rangle \cap \langle gD, U_1/D \rangle$$

is normal in G/D. In other words, every cyclic subgroup of G/D is normal in G/D. Repeating the above arguments, we obtain again that G is abelian, the final contradiction. Hence G is a minimax soluble-by-finite group. Since $\mathrm{Tor}(G)$ is finite, G contains a normal subgroup H of finite index such that $H \cap \mathrm{Tor}(G) = \langle 1 \rangle$ (see for example [5], Corollary 2.4.6]). Then the locally nilpotent radical A of H is torsion-free and nilpotent and H/A is finitely generated and abelian-by-finite (see for example [5], Theorem 6.2.12). An application of Lemma 3.2 shows that A is abelian. □

Lemma 3.9 *Let G be a group and A be a normal torsion-free abelian minimax subgroup. If x is an element of infinite order such that*

$$\langle x \rangle \cap C_G(A) = \langle 1 \rangle,$$

then there exists a set $\{k_n | n \in \mathbb{N}\}$ of positive integers such that the subgroups $\langle A, x^{k_n} \rangle$ are pairwise non-isomorphic.

Proof — Suppose that there exists an infinite set π of primes such that A/A^p is $\langle x \rangle$-central for each $p \in \pi$. In this case

$$[A, x] \leqslant A^p$$

for each $p \in \pi$. Since A is minimax, $Sp(A)$ is finite. Therefore without loss of generality we can assume that $\pi \cap Sp(A) = \emptyset$. Choose in A a finitely generated subgroup D such that A/D is divisible and

$$\Pi(A/D) = Sp(A).$$

Let $p \in \pi$, then D/D^p is a Sylow p-subgroup of A/D^p. It follows that

$$A/D^p = D/D^p \times C_p/D^p.$$

Then $(A/D^p)^p = C_p/D^p$. On the other hand,

$$(A/D^p)^p = A^p D^p/D^p = A^p/D^p.$$

It follows that

$$(A^p/D^p) \cap (D/D^p) = (C_p/D^p) \cap (D/D^p) = \langle 1 \rangle,$$

so $A^p \cap D = D^p$. It follows that

$$D \cap \bigcap_{p \in \pi} A^p = \bigcap_{p \in \pi} (D \cap A^p) = \bigcap_{p \in \pi} D^p = \langle 1 \rangle.$$

Since the factor A/D is periodic and A is torsion-free, we obtain that

$$\bigcap_{p\in\pi} A^p = \langle 1\rangle.$$

The inclusion $[A,x] \leqslant A^p$ for each $p \in \pi$ implies that

$$[A,x] \leqslant \bigcap_{p\in\pi} A^p = \langle 1\rangle,$$

and we obtain a contradiction. This contradiction shows that there exists a set σ of primes such that $\mathcal{P} \setminus \sigma$ is finite and A/A^p is not $\langle x\rangle$-central for each $p \in \sigma$. Let $p_1 \in \sigma$ and $s_1 = |A/A^{p_1}|$. Put $x_1 = x^{(s_1)!}$. Then $[A,x_1] \leqslant A^{p_1}$. By such a choice the subgroups

$$\langle A,x\rangle \quad \text{and} \quad \langle A,x_1\rangle$$

are not isomorphic, because $A^{p_1} \not\geqslant [A,x]$, but $[A,x_1] \leqslant A^{p_1}$. Choose now a prime $p_2 \neq p_1$ such that A/A^{p_2} is not $\langle x_1\rangle$-central. Let $s_2 = |A/A^{p_2}|$. Put $x_2 = x_1^{s_2!}$. Then

$$[A,x_2] \not\leqslant A^{p_1} \quad \text{and} \quad [A,x_2] \leqslant A^{p_2}.$$

By such a choice the subgroups $\langle A,x\rangle$, $\langle A,x_1\rangle$ and $\langle A,x_2\rangle$ are not pairwise isomorphic. Using similar arguments, we choose a subset $\{x_n | n \in \mathbb{N}\}$ of elements of the subgroup $\langle x\rangle$ such that the subgroups $\langle A,x_n\rangle$ are not pairwise isomorphic, $n \in \mathbb{N}$. □

Now we can prove Theorems B and C.

Theorem C *Let* G *be a group in which the family of all subgroups has finite isomorphism type. If* G *is a locally generalized radical group, then* G *contains a normal minimax torsion-free abelian subgroup of finite index.*

Proof — Clearly G does not contain an infinite locally finite subgroup. Let A be a torsion-free abelian subgroup of G. If we suppose that A has infinite 0-rank, then A contains a free abelian subgroup A_0 of countable 0-rank. We have $A_0 = \mathrm{Dr}_{n\in\mathbb{N}}\langle a_n\rangle$, where a_n is an element of infinite order for all $n \in \mathbb{N}$. Since $r_0(\mathrm{Dr}_{1\leqslant n\leqslant k}\langle a_n\rangle) = k$, the subgroups $\mathrm{Dr}_{1\leqslant n\leqslant k}\langle a_n\rangle$ and $\mathrm{Dr}_{1\leqslant n\leqslant m}\langle a_n\rangle$ cannot be isomorphic if $k \neq m$. It follows that all subgroups $\mathrm{Dr}_{1\leqslant n\leqslant k}\langle a_n\rangle$, $k \in \mathbb{N}$ are not pairwise isomorphic, and we obtain a contradiction. This contradiction shows that $r_0(A)$ is finite. Therefore A contains a (finitely generated) free abelian subgroup C such that A/C is periodic, moreover its Sylow p-subgroups are Chernikov for each prime p. Suppose that A is not minimax. Then the set $\Pi(A/C)$ is infinite. Thus there exists in $\Pi(A/C)$ a family $\{\rho_n | n \in \mathbb{N}\}$ of infinite subsets ρ_n such that

$$\bigcup_{n\in\mathbb{N}} \rho_n = \Pi(A/C)$$

and $\rho_n \cap \rho_m = \emptyset$ whenever $n \neq m$. Consider the ascending chain

$$S_0 = C \leqslant S_1 \leqslant \ldots S_n \leqslant S_{n+1} \leqslant \ldots$$

of subgroups, defined by the rule: S_1/C is the Sylow ρ_1-subgroup of A/C, S_2/C is the Sylow $(\rho_1 \cup \rho_2)$-subgroup of A/C, S_n/C is the Sylow $(\rho_1 \cup \ldots \cup \rho_n)$-subgroup of A/C, $n \in \mathbb{N}$. It is not hard to prove that the subgroups S_n and S_{n+k} cannot be isomorphic for all positive integers k, and we obtain a contradiction. This contradiction proves that A is minimax. It turns out that every abelian subgroup of G is minimax. Suppose that G is not (abelian-by-finite). Then Proposition 3.8 shows that G has normal subgroups $A \leqslant K \leqslant G$ where A is an abelian minimax torsion-free subgroup, K/A is an abelian finitely generated torsion-free group and G/K is finite. Since G is not abelian-by-finite, the subgroup K has an element x such that

$$\langle x \rangle \cap C_G(A) = \langle 1 \rangle.$$

But in this case Lemma 3.9 shows that there exists a set $\{k_n | n \in \mathbb{N}\}$ of positive integers such that the subgroups $\langle A, x^{k_n} \rangle$ are pairwise non-isomorphic, and we again obtain a contradiction. This contradiction shows that G is abelian-by-finite. □

Theorem B *Let G be a locally generalized radical group. If $G \in \mathcal{C}$, then G is minimax and abelian-by-finite.*

Proof — If G has an infinite locally finite subgroup, the result follows from Theorem 3.3 and Theorem A. Suppose that every periodic subgroup of G is finite and that G does not contain a normal abelian subgroup of finite index. Using Proposition 3.8 we obtain that G has normal subgroups $A \leqslant K \leqslant G$ where A is an abelian minimax torsion-free subgroup, K/A is an abelian finitely generated torsion-free group, G/K is finite. Let

$$K/A = \langle y_1 A \rangle \times \ldots \times \langle y_n A \rangle.$$

If $\langle y_j\rangle \cap C_K(A) \neq \langle 1\rangle$ for every $j, 1 \leqslant j \leqslant n$, then the index $|K : C_K(A)|$ is finite. Being nilpotent and torsion-free, the subgroup $C_K(A)$ must be abelian by Lemma 3.2. But in this case G is abelian-by-finite, and we obtain a contradiction. This contradiction shows that there is a number j such that $\langle y_j\rangle \cap C_K(A) = \langle 1\rangle$. Without loss of generality we may assume that $\langle y_1\rangle \cap C_K(A) = \langle 1\rangle$. Using the arguments from the proof of Lemma 3.9 we can find a sequence of primes $\{p_n | n \in \mathbb{N}\}$ and a subset $\{x_n | n \in \mathbb{N}\}$ of the subgroup $\langle y_1\rangle$ such that:

$$[A, x_1] \leqslant A^{p_1}, \text{ but } A^{p_2} \text{ does not contain } [A, x_1],$$
$$[A, x_2] \leqslant A^{p_1}, [A, x_2] \leqslant A^{p_2}, \text{ but } A^{p_3} \text{ does not contain } [A, x_2],$$
$$\dots$$
$$[A, x_n] \leqslant A^{p_1}, \dots, [A, x_n] \leqslant A^{p_n}, \text{ but } A^{p_{n+1}} \text{ does not contain } [A, x_n].$$

Put now $X_n = \langle A, x_n\rangle = A\langle x_n\rangle$. Clearly $[X_n, X_n] = [A, x_n]$. The equality $\langle 1\rangle = \langle y_1\rangle \cap C_K(A)$ and Lemma 3.2 imply that the locally nilpotent radical $L\mathcal{N}(X_n)$ coincides with A. Hence we have:

$$[X_n, X_n] \leqslant L\mathcal{N}(X_n)^{p_1}, [X_n, X_n] \leqslant L\mathcal{N}(X_n)^{p_2}, \dots, [X_n, X_n] \leqslant L\mathcal{N}(X_n)^{p_n},$$

but $L\mathcal{N}(X_n)^{p_{n+1}}$ does not contain $[X_n, X_n]$.

It follows that the subgroups X_n, X_k cannot be isomorphic whenever $n < k$. Let t be a positive integer. Consider now the subgroup

$$Y_{t,n} = \langle A^t, x_n\rangle = A^t\langle x_n\rangle.$$

Using the above arguments we obtain that $L\mathcal{N}(\langle A^t, x_n\rangle) = A^t$. Also

$$[Y_{t,n}, Y_{t,n}] = [A^t, x_n] = [A, x_n]^t.$$

Now we obtain

$$\begin{aligned}[Y_{t,n}, Y_{t,n}] &= [X_n, X_n]^t \leqslant \left(L\mathcal{N}(X_n)^{p_j}\right)^t \\ &= \left(L\mathcal{N}(X_n)^t\right)^{p_j} = (A^t)^{p_j} = L\mathcal{N}(Y_{t,n})^{p_j},\end{aligned}$$

for $j \in \{1, \dots, n\}$, $j \leqslant n$. The mapping

$$f : A \to A,$$

defined by the rule $f(a) = a^t, a \in A$, is an $\langle x_n\rangle$-endomorphism, because A is abelian. Since A is torsion-free, f is a monomorphism. Then the fact that

$$A^{p_{n+1}} = L\mathcal{N}(X_n)^{p_{n+1}}$$

does not contain $[X_n, X_n]$ implies that

$$\begin{aligned}(A^t)^{p_{n+1}} &= (A^{p_{n+1}})^t = \big(L\mathcal{N}(X_n)^{p_{n+1}}\big)^t \\ &= \big(L\mathcal{N}(X_n)^t\big)^{p_{n+1}} = L\mathcal{N}(Y_{t,n})^{p_{n+1}}\end{aligned}$$

does not contain $[Y_{t,n}, Y_{t,n}]$. It follows that the subgroups $\langle A^t, x_n\rangle$ and $\langle A^t, x_k\rangle$ cannot be isomorphic whenever $n < k$, $n, k \in \mathbb{N}$. Put now $Z_n = \langle A^{p_{n+1}}, x_n\rangle$, $n \in \mathbb{N}$. Suppose that Z_n is normal in G. It follows that $[A, x_n] \leqslant Z_n$. On the other hand, $[A, x_n] \leqslant A$, so that

$$[A, x_n] \leqslant A \cap Z_n = A^{p_{n+1}},$$

and we obtain a contradiction. This contradiction shows that the subgroup Z_n is not normal in G. This is true for every $n \in \mathbb{N}$. But we have proved above that the subgroups Z_n and Z_k cannot be isomorphic whenever $n < k$. This final contradiction proves the result. □

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