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Finite Groups with Partially σ-Subnormal Subgroups in Short Maximal Chains

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Abstract

Throughout this paper, all groups are finite and G always denotes a finite group.

Let $\sigma = \{\sigma_i \mid i \in I\}$ be a partition of the set of all primes $\mathbb{P}$. The group G is said to be: *σ-primary* if G is a σ_i-group for some $i = i(G)$; *σ-soluble* if every chief factor of G is σ-primary. A subgroup A of G is called: *σ-subnormal* in G if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$; $\mathfrak{U}$-*normal* in G if either A is normal in G or $A_G \neq A^G$ and every chief factor of G between A_G and A^G is cyclic. We say that a subgroup A of G is *partially σ-subnormal* in G if $A = \langle L, T \rangle$, where L is $\mathfrak{U}$-normal and T is σ-subnormal subgroups of G.

In this paper, we prove that if in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2, or M_1 is partially σ-subnormal in G, then G is σ-soluble.

Some known results are generalized.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes and $\sigma = \{\sigma_i \mid i \in I \subseteq \mathbb{N}\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n. As usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G; $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

A subgroup A of G is said to be $\mathfrak{U}$-*normal* in G if either A is normal in G, or $A_G \neq A^G$ and every chief factor of G between A_G and A^G is cyclic (see [7]).

Recall some concepts of the papers [16, 17, 19] which play a fundamental role in the theory of σ-properties of groups. A group G is said to be: σ-*primary* if G is a σ_i-group for some $i = i(G)$; σ-*nilpotent* if $G = G_1 \times \ldots \times G_t$ for some σ-primary groups $G_1, \ldots, G_t$; σ-*soluble* if every chief factor of G is σ-primary. A subgroup A of G is called σ-*subnormal* in G if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_t = G$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$. Note, in passing, that the σ-subnormal subgroups proved to be very useful and found many applications in the study of various classes of generalized solvable groups (see, for example, [1]–[3],[8],[11],[12],[16]–[19]).

Now, recall that if

$$M_n < M_{n-1} < \ldots < M_1 < M_0 = G, \tag{$*$}$$

where M_i is a maximal subgroup of M_{i-1} for all $i = 1, \ldots, n$, then the chain $(*)$ is said to be a *maximal chain of* G *of length* n and M_n $(n > 0)$, is an n-*maximal subgroup* of G.

The relationship between n-maximal subgroups (where $n > 1$) of G and the structure of G was studied by many authors. One of the earliest results in this line research was obtained by Huppert in the article [9] who established the supersolubility of the group whose all second maximal subgroups are normal. In the same article Huppert proved also that if all 3-maximal subgroups of G are normal in G, then G is soluble. These two results were developed by many authors. Spencer studied [20] the groups G whose every n-maximal chain includes at least one proper subnormal subgroup of G and he proved that G is soluble if in every maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2, or M_1 is subnormal in G. The σ-generalization of the last result was obtained in [6]. The solubility of groups in which all 3-maximal subgroups are modular was proved in [13].

In this paper, we obtain generalizations of some of these results on the base of the following definition.

Definition 1.1 *We say that a subgroup* A *of* G *is partially σ-subnormal in* G *if* $A = \langle L, T\rangle$, *where* L *is* $\mathfrak{U}$*-normal and* T *is σ-subnormal subgroups of* G.

It is clear that all $\mathfrak{U}$-normal and all σ-subnormal subgroups are partially σ-subnormal.

Now consider the following example.

Example 1.2 Let p, q, r, t be distinct primes, where q divides $p-1$ and t divides $r-1$, and let $\sigma = \{\{t\}, \{t\}'\}$, where $\{t\}'$ is the set of all primes $s \neq t$. Let $V = Q \rtimes C_p$, where Q is a simple $\mathbb{F}_q C_p$-module which is faithful for C_p, and $C_r \rtimes C_t$ a non-abelian group of order rt. Let $G = V \times (C_r \rtimes C_t)$. Then $C_t^G = C_r \rtimes C_t$, so C_t is $\mathfrak{U}$-normal in G. Let B be a subgroup of order q in Q. Then $B < Q$ since $p > q$ and the subgroup $H = \langle C_t, B\rangle$ is partially σ-subnormal in G.

Assume that H is $\mathfrak{U}$-normal in G. Then $B = H \cap V$ is $\mathfrak{U}$-normal in V by Lemma 2.8 (5) below. Hence Q is cyclic since $B^G = Q$ and $B_G = 1$. This contradiction shows that H is not $\mathfrak{U}$-normal in G.

Similarly, if H is σ-subnormal in G, then

$$C_t = H \cap (C_r \rtimes C_t)$$

is σ-subnormal in $C_r \rtimes C_t$ and so C_t is normal in $C_r \rtimes C_t$ by Lemma 2.10 (1),(5) below. But then $C_r \rtimes C_t$ is abelian. This contradiction shows that H is not σ-subnormal in G.

Our main goal here is to prove the following theorem.

Theorem 1.3 *If in every maximal chain* $M_3 < M_2 < M_1 < M_0 = G$ *of* G *of length* 3 *at least one of the subgroups* M_3, M_2, *or* M_1 *is partially σ-subnormal in* G, *then* G *is σ-soluble.*

Corollary 1.4 (Spencer [20]) *If in every maximal chain*

$$M_3 < M_2 < M_1 < M_0 = G$$

of G *of length* 3 *at least one of the subgroups* M_3, M_2, *or* M_1 *is subnormal in* G, *then* G *is soluble.*

Corollary 1.5 (Huppert [9]) *If every* 3*-maximal subgroup of* G *is normal in* G, *then* G *is soluble.*

Corollary 1.6 (Guo and Skiba [6]) *If in every maximal chain*

$$M_3 < M_2 < M_1 < M_0 = G$$

of G *of length* 3 *at least one of the subgroups* M_3*,* M_2*, or* M_1 *is* σ*-subnormal in* G*, then* G *is* σ*-soluble.*

Recall that a subgroup M of G is called *modular* if M is a modular element (in the sense of Kurosh [14, 2, p. 43]) of the lattice $\mathcal{L}(G)$ of all subgroups of G, that is, (i) $\langle X, M\cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X \leqslant G, Z \leqslant G$ such that $X \leqslant Z$, and (ii) $\langle M, Y\cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y \leqslant G, Z \leqslant G$ such that $M \leqslant Z$.

Recall also that from Theorem 5.2.5 in [14] it follows that every modular subgroup is $\mathfrak{U}$-normal. Therefore, we get from Theorem 1.3 also the following result.

Corollary 1.7 *If in every maximal chain* $M_3 < M_2 < M_1 < M_0 = G$ *of* G *of length* 3 *one of* M_3*,* M_2 *and* M_1 *is modular in* G*, then* G *is soluble.*

Hence, from Corollary 1.7 we get the following known fact.

Corollary 1.8 (Schmidt [13]) *If every* 3*-maximal subgroup of* G *is modular in* G*, then* G *is soluble.*

2 Preliminaries

If $K \trianglelefteq H \leqslant G$, then H/K is called a *section* of G; such a section is called *normal* if $K, H \trianglelefteq G$. We call any set Σ of normal sections of G a *stratification* of G [19, 18] provided: (i) Σ *is* G*-closed*, that is, $H/K \in \Sigma$ whenever $H/K \simeq_G T/L \in \Sigma$, and (ii) $L/K, H/L \in \Sigma$ for each triple $K < L < H$, where $H/K \in \Sigma$ and $L \trianglelefteq G$.

If Δ is any (may be empty) set of normal sections of G, then we use $\Sigma_G(\Delta)$ to denote the set of all normal sections T/L of G such that either $L = T$ or there exists a series

$$L = L_0 \leqslant L_1 \leqslant \ldots \leqslant L_{t-1} \leqslant L_t = T,$$

where $L_i/L_{i-1} \in \Delta$ for all $i = 1, \ldots, t$ (see [19]).

Now let Σ be any stratification of G. Then we write $\mathcal{L}_\Sigma(G)$ to denote the set of all subgroups A of G such that $A^G/A_G \in \Sigma$ (see [19],[18]).

The following lemma is evident.

Lemma 2.1 *Let Δ be a set of chief factors of G such that $H/K \in \Delta$ whenever $H/K \simeq_G T/L \in \Delta$. Then Δ is a stratification of G.*

Note that if H/K and T/L are G-isomorphic chief factors of G, then H/K is cyclic if and only if T/L is cyclic. Therefore we get from Lemma 2.1 the following consequence.

Corollary 2.2 *Let Δ be the set of all cyclic chief factors of G. Then Δ is a stratification of G.*

We use $\mathcal{L}(G)$ to denote the lattice of all subgroups of G.

Lemma 2.3 (see [19], Theorem 1.4) *If $\Sigma = \Sigma_G(\Delta)$ for some stratification Δ of G, then $\mathcal{L}_\Sigma(G)$ is a sublattice of $\mathcal{L}(G)$.*

From Corollary 2.2 and Lemma 2.3 we get the following result.

Corollary 2.4 *The set of all $\mathfrak{U}$-normal subgroups of G forms a sublattice of the lattice $\mathcal{L}(G)$.*

Let N be a normal subgroup of G. Then for any stratification Δ of G we use $\Delta N/N$ to denote the set $\{(NH/N)/(NK/N) \mid H/K \in \Delta\}$.

Lemma 2.5 (see [19], Lemmas 2.1 and 3.2) *Let $N \trianglelefteq G$ and let Δ be a stratification of G. Then:*

(1) $\Delta N/N$ *is a stratification of* G/N *and*

$$\Delta N/N = \{(H/N)/(K/N) \mid H/K \in \Delta \ \text{and}\ N \leqslant K\};$$

(2) $\Sigma_G(\Delta)N/N = \Sigma_{G/N}(\Delta N/N)$.

Lemma 2.6 (see [19], Lemma 3.1) *Let Σ be a stratification of G and let $A \in \mathcal{L}_\Sigma(G)$ and $N \leqslant H \leqslant G$, where $N \trianglelefteq G$. Then*

(1) $AN/N \in \mathcal{L}_{\Sigma N/N}(G/N)$, *and*

(2) *if* $H/N \in \mathcal{L}_{\Sigma N/N}(G/N)$, *then* $H \in \mathcal{L}_\Sigma(G)$.

We use $Z_{\mathfrak{U}}(G)$ to denote the product of all normal subgroups N of G such that every chief factor of G below N is cyclic.

Lemma 2.7 *Let $Z = Z_{\mathfrak{U}}(G)$. Then every chief factor of G below Z is cyclic. Moreover,*

(1) $Z \cap E \leqslant Z_{\mathfrak{U}}(E)$ *for every subgroup* E *of* G, *and*

(2) $NZ/N \leqslant Z_{\mathfrak{U}}(G/N)$ *for every normal subgroup* N *of* G.

Proof — First we show that every chief factor of G below Z is cyclic. In fact, it is enough to show that if A and B are normal subgroups of G such that all chief factors of G below A and all chief factors of G below B are cyclic, then each chief factor H/K of G below AB is also cyclic. Moreover, in view of the Jordan-Hölder theorem for the chief series, it is enough to show that if $A \leqslant K < H \leqslant AB$, then H/K is cyclic. But the latest fact follows from

$$H/K = A(H \cap B)/K = K(H \cap B)/K$$

and the G-isomorphism $K(H \cap B)/K \simeq (H \cap B)/(K \cap B)$. Hence every chief factor of G below Z is cyclic.

(1) Let

$$1 = Z_0 < Z_1 < \ldots < Z_{t-1} < Z_t = Z$$

be a chief series of G below Z. Then every factor Z_i/Z_{i-1} of this series is cyclic, that is, $|Z_i/Z_{i-1}|$ is a prime. Now consider the normal series

$$1 = Z_0 \cap E \leqslant Z_1 \cap E \leqslant \ldots \leqslant Z_{t-1} \cap E \leqslant Z_t \cap E = Z \cap E \qquad (*)$$

in E. Assume that $Z_i \cap E \neq Z_{i-1} \cap E$. Then from the isomorphism

$$(Z_i \cap E)/(Z_{i-1} \cap E) \simeq (Z_i \cap E)Z_{i-1}/Z_{i-1} = Z_i/Z_{i-1}$$

we get that $|(Z_i \cap E)/(Z_{i-1} \cap E)|$ is a prime, so every non-trivial factor $(Z_i \cap E)/(Z_{i-1} \cap E)$ of the series $(*)$ is cyclic. Therefore, in view of the Jordan-Hölder theorem for the chief series, every chief factor of E below $Z \cap E$ is cyclic. Hence $Z \cap E \leqslant Z_{\mathfrak{U}}(E)$.

(2) Let $(H/N)/(K/N)$ be any chief factor of G/N such that $H \leqslant NZ$. Then $H/K = (H \cap Z)K/K$ is a chief factor of G. On the other hand, from the G-isomorphism

$$(H \cap Z)K/K \simeq (H \cap Z)/(K \cap Z)$$

it follows that H/K is cyclic since every chief factor of G below Z is cyclic. Thus $(H/N)/(K/N)$ cyclic, so $NZ/N \leqslant Z_{\mathfrak{U}}(G/N)$. □

Lemma 2.8 *Let* A, B *and* N *be subgroups of* G, *where* A *is* $\mathfrak{U}$*-normal and* N *is normal in* G.

(1) *If* B *is* $\mathfrak{U}$*-normal in* G, *then* $\langle A, B\rangle$ *is* $\mathfrak{U}$*-normal in* G.

(2) AN/N *is* $\mathfrak{U}$*-normal in* G/N.

(3) *If* $N \leqslant B$ *and* B/N *is* $\mathfrak{U}$*-normal in* G*, then* B *is* $\mathfrak{U}$*-normal in* G.

(4) N *is* $\mathfrak{U}$*-normal in* G.

(5) $A \cap E$ *is* $\mathfrak{U}$*-normal in* E *for all subgroups* E *of* G.

(6) *If* φ *is an isomorphism of* G *onto* $\overline{G}$*, then* A^{φ} *is* $\mathfrak{U}$*-normal in* $\overline{G}$.

(7) *A maximal subgroup* M *of* G *is* $\mathfrak{U}$*-normal in* G *if and only if* G/M_G *is supersoluble.*

Proof — Let Δ be the set of all cyclic chief factors of G and $\Sigma = \Sigma_G(\Delta)$. Let $\mathcal{L}$ be the set of all $\mathfrak{U}$-normal subgroups of G. Then $\mathcal{L} = \mathcal{L}_{\Sigma}(G)$.

(1) This follows from Corollary 2.4.

(2) By Lemma 2.6 (1), $AN/N \in \mathcal{L}_{\Sigma N/N}(G/N)$. On the other hand,

$$\Sigma N/N = \Sigma_0 := \Sigma_{G/N}(\Delta N/N)$$

by Lemma 2.5 (2). Finally, note that $\Delta N/N$ is the set (maybe empty) of all cyclic chief factors of G/N by Lemma 2.5 (1). Hence $\mathcal{L}_{\Sigma_0}(G/N)$ is the set of all $\mathfrak{U}$-normal subgroups of G/N. Hence AN/N is $\mathfrak{U}$-normal in G/N.

(3) Since $B/N \in \mathcal{L}_{\Sigma_0}(G/N)$, where

$$\Sigma_0 := \Sigma_{G/N}(\Delta N/N) \text{ and } \Delta N/N = \{(H/N)/(K/N) \mid H/K \in \Delta \text{ and } N \leqslant K\}$$

is the set of all cyclic chief factors of G/N, then either B is normal in G or $B_G \neq B^G$ and every chief factor of G between B_G and B^G is cyclic. Hence B is $\mathfrak{U}$-normal in G.

(4) This follows from Definition 1.1.

(5) First note that

$$(A^G/A_G) \cap (EA_G/A_G) = A_G(A^G \cap E)/A_G \leqslant Z_{\mathfrak{U}}(EA_G/A_G)$$

by Lemma 2.7 (1) since by hypothesis we have $A^G/A_G \leqslant Z_{\mathfrak{U}}(G/A_G)$. On the other hand, we have

$$f(Z_{\mathfrak{U}}(EA_G/A_G)) = Z_{\mathfrak{U}}(E/(A_G \cap E)),$$

where $f : EA_G/A_G \to E/(E \cap A_G)$ is the canonical isomorphism from EA_G/A_G onto $E/(E \cap A_G)$. Hence

$$f(A_G(A^G \cap E)/A_G) = (A^G \cap E)/(A_G \cap E) \leqslant Z_{\mathfrak{U}}(E/(A_G \cap E)),$$

where

$$A_G \cap E \leqslant (E \cap A)_E \leqslant A \cap E \leqslant (A \cap E)^E \leqslant A^G \cap E,$$

and so

$$(A \cap E)^E/(A \cap E)_E \leqslant Z_{\mathfrak{U}}(E/(A \cap E)_E))$$

by Lemma 2.7. Hence $A \cap E$ is $\mathfrak{U}$-normal in E.

(6) This assertion is evident.

(7) First assume that M is $\mathfrak{U}$-normal in G, that is, either M is normal in G or $M_G \neq M^G$ and every chief factor of G between M_G and M^G is cyclic. Then the maximality of M implies that every chief factor of G between M_G and G is cyclic, so G/M_G is supersoluble.

Finally, if G/M_G is a supersoluble, then M is evidently $\mathfrak{U}$-normal in G. □

Lemma 2.9 (see Corollary 2.4 and Lemma 2.5 of [16]) *The class of all σ-nilpotent groups* $\mathfrak{N}_\sigma$ *is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if* H *is a normal subgroup of* G *and* $H/(H \cap \Phi(G))$ *is σ-nilpotent, then* H *is σ-nilpotent.*

Recall that $G^{\mathfrak{N}}$ denotes the *σ-nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Lemma 2.10 *Let* A, K *and* N *be subgroups of* G, *where* A *is σ-subnormal in* G *and* N *is normal in* G. *Then the following statements hold.*

(1) $A \cap K$ *is σ-subnormal in* K.

(2) AN/N *is σ-subnormal in* G/N.

(3) *If* $N \leqslant K$ *and* K/N *is σ-subnormal in* G/N, *then* K *is σ-subnormal in* G.

(4) *If* K *is σ-subnormal in* G, *then* $A \cap K$ *and* $\langle A, K\rangle$ *are σ-subnormal in* G.

(5) *If* A *is a* σ_i*-group, then* $A \leqslant O_{\sigma_i}(G)$.

(6) $A^{\mathfrak{N}_\sigma}$ *is subnormal in* G.

Proof — (1)–(5) See Lemma 2.6 of [16].

(6) Assume that this assertion is false and let G be a counterexample of minimal order. By hypothesis, there is a chain

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_r = G$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $M = A_{r-1}$. We can assume without loss of generality that $M \neq G$.

First we show that $A^{\mathfrak{N}_\sigma} \leqslant M_G$. This is clear if M is normal in G. Now assume that G/M_G is a σ_i-group for some i. Then $G^{\mathfrak{N}_\sigma} \leqslant M_G$. Moreover, from the isomorphism $AG^{\mathfrak{N}_\sigma}/G^{\mathfrak{N}_\sigma} \simeq A/(A \cap G^{\mathfrak{N}_\sigma})$ and Lemma 2.9 we get that $A^{\mathfrak{N}_\sigma} \leqslant G^{\mathfrak{N}_\sigma} \leqslant M_G$. The choice of G implies that $A^{\mathfrak{N}_\sigma}$ is subnormal in AM_G, so $A^{\mathfrak{N}_\sigma}$ is subnormal in M_G. Therefore $A^{\mathfrak{N}_\sigma}$ is subnormal in G. □

Lemma 2.11 *Let* A, B *and* N *be subgroups of* G, *where* A *is partially σ-subnormal and* N *is normal in* G.

(1) AN/N *is partially σ-subnormal in* G/N.

(2) *If* $A \leqslant B$, *then* A *is partially σ-subnormal in* B.

(3) *If* $N \leqslant B$ *and* B/N *is partially σ-subnormal in* G/N, *then* B *is partially σ-subnormal in* G.

(4) *If* B *is partially σ-subnormal in* G, *then* $\langle A, B\rangle$ *is partially σ-subnormal in* G.

(5) *If* A *is maximal in* G, *then* G/A_G *is supersoluble or σ-primary.*

Proof — Let $A = \langle L, T\rangle$, where L is $\mathfrak{U}$-normal and T is σ-subnormal subgroups of G.

(1) $AN/N = \langle LN/N, TN/N\rangle$, where TN/N is σ-subnormal in G/N by Lemma 2.10 (2) and LN/N is $\mathfrak{U}$-normal in G/N by Lemma 2.8 (2). Hence AN/N is partially σ-subnormal in G/N.

(2) L is a $\mathfrak{U}$-normal subgroup of B by Lemma 2.8 (5) and T is a σ-subnormal subgroup of B by Lemma 2.10 (1). Hence $A = \langle L, T\rangle$ is partially σ-subnormal in B.

(3) Let $B/N = \langle V/N, W/N \rangle$, where V/N is $\mathfrak{U}$-normal in G/N and W/N is σ-subnormal in G/N. Then $B = \langle V, W \rangle$, where V is $\mathfrak{U}$-normal in G by Lemma 2.8 (3) and W is σ-subnormal in G by Lemma 2.10 (3), so B is partially σ-subnormal in G.

(4) Let $B = \langle V, W \rangle$, where V is $\mathfrak{U}$-normal and W is σ-subnormal subgroups of G. Then

$$\langle A, B \rangle = \langle \langle L, T \rangle, \langle V, W \rangle \rangle = \langle \langle L, V \rangle, \langle T, W \rangle \rangle,$$

where $\langle L, V \rangle$ is $\mathfrak{U}$-normal in G by Lemma 2.8 (1) and $\langle T, W \rangle$ is σ-subnormal in G by Lemma 2.10 (4). Hence $\langle A, B \rangle$ is partially σ-subnormal in G.

(5) If A is normal in G, it is evident. Now suppose that $A \neq A^G$. Then A/A_G is a partially σ-subnormal maximal subgroup of G/A_G by Part (1). Hence we can assume without loss of generality that $A_G = 1$ and so G is primitive. Hence, in view of [4, Ch. A, Theorem 15.2], either G has a unique minimal normal subgroup V and $C_G(V) \leqslant V$ or G has the only two different minimal normal subgroups V and W which are non-abelian and for which we have $V \simeq W \simeq VW \cap A$, $G = V \rtimes A = W \rtimes A$, $V = C_G(W)$ and $W = C_G(V)$.

First suppose that $L^G \neq 1$. In this case we can assume without loss of generality that $V \leqslant L^G$. Then from $L_G \leqslant A_G = 1$ we get that

$$V \leqslant L^G \leqslant Z_{\mathfrak{U}}(G)$$

by Lemma 2.7 and so V is cyclic. Then V is abelian and so, in fact, $C_G(V) = V$ is a cyclic group of prime order. Hence $G/V = G/C_G(V)$ is cyclic and so $G \simeq G/1 = G/A_G$ is supersoluble.

Now suppose that $L^G = 1$, so $A = T$. In view of Lemma 2.10 (6), $T^{\mathfrak{N}_\sigma} = A^{\mathfrak{N}_\sigma}$ is subnormal in G. Then, in view of [4, Ch. A, Lemma 14.3],

$$(A^{\mathfrak{N}_\sigma})^G = (A^{\mathfrak{N}_\sigma})^{RA} = (A^{\mathfrak{N}_\sigma})^A \leqslant A_G = 1.$$

Hence $A = T$ is a σ-nilpotent σ-subnormal subgroup of G. Hence, in view of $A \neq A^G$, $A^G = G$ is σ-nilpotent by Lemma 2.10 (5) and so, in fact, $G \simeq G/1 = G/A_G$ is σ-primary since A is a maximal subgroup of G. Hence (5) holds. $\square$

Recall that a *Schmidt group* is a non-nilpotent group in which all proper subgroups are nilpotent.

Lemma 2.12 (see [15], VI, Theorem 26.1) *If* G *is a Schmidt group, then* $G = P \rtimes Q$*, where* $P = G^{\mathfrak{N}}$ *is a Sylow* p*-subgroup of* G *and* $Q = \langle x \rangle$ *is a cyclic Sylow* q*-subgroup of* G *for some primes* $p \neq q$*.*

3 Proof of Theorem 1.3

Suppose that this theorem is false and let G be a counterexample of minimal order.

1) *The group* G/R *is σ-soluble for every minimal normal subgroup* R *of* G*. Hence* R *is the unique minimal normal subgroup of* G *and* R *is not σ-primary.*

First we show that G/R is σ-soluble for every minimal normal subgroup R of G. Assume that this is false. Then G/R is not nilpotent, so G/R has a Schmidt subgroup H/R. Then H/R is soluble by Lemma 2.12, so $H < G$. Moreover, from Lemma 2.12 it follows that for every prime p dividing $|H/R|$ and for every Sylow p-subgroup P of H/R it follows that P is contained in some 2-maximal subgroup of G/R. Hence R is contained in some 3-maximal subgroup of G. Now let

$$M_3/R < M_2/R < M_1/R < M_0/R = G/R$$

be an arbitrary maximal chain of G/R of length 3. Then

$$M_3 < M_2 < M_1 < M_0 = G$$

is a maximal chain in G of length 3 and so for some $i > 0$ the subgroup M_i partially σ-subnormal in G by hypothesis. But then M_i/R is partially σ-subnormal in G/R by Lemma 2.11 (1). Therefore the hypothesis holds for G/R, so the choice of G implies that G/R is σ-soluble, a contradiction.

Hence G/N is σ-soluble for every minimal normal subgroup N of G. Moreover, if $N \neq R$, then from the G-isomorphism $RN/N \simeq R$ we get that R is σ-primary and so G is σ-soluble, contrary to the choice of G. Therefore R is the unique minimal normal subgroup of G and R is not σ-primary.

From Claim (1) it follows that R is not abelian. Let p be any odd prime dividing $|R|$ and R_p a Sylow p-subgroup of R. Let G_p be a Sylow p-subgroup of G such that $R_p = G_p \cap R$. Then $G_p \leqslant N_G(R_p)$. Moreover, the Frattini argument implies that $G = RN_G(R_p)$. Hence there is a maximal subgroup M of G such that $G_p \leqslant N_G(R_p) \leqslant M$ and $G = RM$. Then $M \neq M_G = 1$ by Claim (1).

2) *The subgroup* M *is not partially* σ*-subnormal in* G.

Assume that this is false and let $M = \langle A, B\rangle$, where A is $\mathfrak{U}$-normal and B is σ-subnormal subgroups of G.

Suppose that $A = 1$, that is, $M = B$ is a σ-subnormal subgroup of G. Then there is a subgroup chain

$$M = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_r = G$$

such that either $M_{i-1} \trianglelefteq M_i$ or $M_i/(M_{i-1})_{M_i}$ is σ-primary for all $i = 1, \ldots, r$, where $M_{r-1} < G$. But M is a maximal subgroup of G and so, in fact, $M = M_{r-1}$, where $M_{r-1} \neq M_G = (M_{r-1})_G = 1$. Hence $G \simeq G/1 = G/M_G$ is σ-primary, so G is σ-soluble.

This contradiction shows that $A \neq 1$. On the other hand,

$$A_G \leqslant M_G = 1 \quad \text{and} \quad A^G/A_G \leqslant Z_{\mathfrak{U}}(G/A_G).$$

Hence $R \leqslant A^G \leqslant Z_{\mathfrak{U}}(G)$ by Claim (1). But then R is abelian since every chief factor of G below $Z_{\mathfrak{U}}(G)$ is cyclic by Lemma 2.7. This contradiction completes the proof of the claim.

3) *The subgroup* $D = M \cap R$ *is not nilpotent. Hence* $D \nleq \Phi(M)$ *and* $|D|$ *is not a prime power.*

Assume that D is nilpotent. Note that $R_p = G_p \cap R \leqslant M \cap R = D$, so R_p a Sylow p-subgroup of D. Then R_p is characteristic in D and so it is normal in M. Hence $Z(J(R_p))$ is normal in M. Since $M_G = 1$, it follows that $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that R is p-nilpotent by Glauberman-Thompson's theorem on the normal p-complements [5, Ch. 8, Theorem 3.1]. But then R is a p-group, contrary to Claim (1). Hence we have (3).

4) $R < G$.

Suppose that $R = G$ is a simple non-abelian group. Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$, and let L be a maximal subgroup of G containing P. Then, in view of [10, IV, Satz 2.8], $|P| > p$. Let V be a maximal subgroup of P.

If $|V| = p$, then P is abelian, so $1 < V < P < L$ by [10, IV, Theorem 7.4] since G is not soluble. On the other hand, in the case when $|V| > p$ we have

$$1 < W < V < P < G,$$

where W is a maximal subgroup of V. Hence there is a 3-maximal subgroup E of G such that $E \neq 1$. But then some proper non-identity subgroup H of G is partially σ-subnormal in G by hypothesis. Hence $H = \langle A, B \rangle$ for some $\mathfrak{U}$-normal subgroup A and σ-subnormal subgroup B of G. Assume that $A \neq 1$. Then from $A_G \leqslant M_G = 1$ we get that $R \leqslant A^G = G \leqslant Z_{\mathfrak{U}}(G)$. Therefore R is abelian by Lemma 2.7, contrary to Claim (1).

Therefore $A = 1$, so $H = B$ is σ-subnormal in G. Then there is a subgroup chain

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G$$

such that either $H_{i-1} \trianglelefteq H_i$ or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all i in $\{1, \ldots, n\}$. It is possible to assume without loss of generality that $V = H_{n-1} < G$. Then $V_G = 1$ since $G = R$ is simple, so $G \simeq G/1$ is σ-primary. This contradiction shows that we have (4).

5) M *is σ-soluble.*

If some maximal subgroup of M has prime order, then M is soluble by [10, IV, Satz 7.4].

Now let $1 < L < T < M$, where L is a maximal subgroup of T and T is a maximal subgroup of M. Since M is not partially σ-subnormal in G by Claim (2), either L or T is partially σ-subnormal in G and so it is partially σ-subnormal in M by Lemma 2.11 (2). Hence the hypothesis holds for M, so M is σ-soluble by the choice of G.

6) $M = D \rtimes T$, *where* T *is a maximal subgroup of* M *of prime order.*

In view of Claim (3), there is a maximal subgroup T of M such that $M = DT$. Then

$$G = RM = R(DT) = RT$$

and so, in view of Claim (4), $T \neq 1$. Assume that $|T|$ is not a prime and let V be a maximal subgroup of T. Then $V \neq 1$. Since M is not partially σ-subnormal in G, at least one of the subgroups T or V is partially σ-subnormal in G by hypothesis. Claim (5) implies that both subgroups V and T are σ-soluble. Consider, for example, the case when V is partially σ-subnormal in G, that is, $V = \langle A, B \rangle$ for some $\mathfrak{U}$-normal subgroup A and some σ-subnormal subgroup B of G. Note that B is also σ-soluble, so in the case when $B \neq 1$ we get that $O_{\sigma_i}(B) \neq 1$ for some i. But $O_{\sigma_i}(B) \leqslant O_{\sigma_i}(G)$ by Lemma 2.10 (5), so $O_{\sigma_i}(G) \neq 1$, which implies that R is σ-primary by Claim (1), a contradiction.

Therefore $B = 1$, that is, $V = A$ is $\mathfrak{U}$-normal in G. It is clear that $A_G = 1$ and hence $1 < A^G = V^G \leqslant Z_{\mathfrak{U}}(G)$, which implies that $R \leqslant Z_{\mathfrak{U}}(G)$. But then R is abelian, a contradiction. Hence $|T|$ is a prime, so $M = D \rtimes T$.

Final contradiction. Since T is a maximal subgroup of M and it is cyclic, M is soluble by [10, IV, Theorem 7.4] and so $|D|$ is a prime power, contrary to Claim (3).

The theorem is proved. □

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