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On Groups of Commutator Symmetries

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Abstract

It is proved that certain subsets of a group, defined by suitable commutator identities, are characteristic subgroups.

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1 Introduction

Let G be a group and let S_n denote the permutation group on the set $\{1, 2, \dots, n\}$, $n > 1$. Let $\pi \in S_n$ and let $x = (x_1, x_2, \dots, x_n)$ be an n-tuple of elements $x_i \in G$. Let $x\pi = y = (y_1, y_2, \dots, y_n)$ be the positional permutation, defined by π, which takes $x_i \in x$ to y_k and $k = i\pi$. The concept of *permutation extension* is introduced in Section 4. For this purpose the positional form of π is preferable to its standard form.

In this paper we consider the following subsets of G:

$$\pi G = \big\{ a \in G : [a, x_1, x_2, \dots, x_n] = [a, y_1, y_2, \dots, y_n] :$$
$$\text{for all } x, y \text{ such that } x\pi = y \big\}$$

and

$$\bar{\pi}G = \big\{a \in G : [a, x_1, x_2, \ldots, x_n] = [a, y_1, y_2, \ldots, y_n]^{-1} : \\ \text{for all } x, y \text{ such that } x\pi = y\big\}.$$

For a subgroup K of S_n, we define $KG = \bigcap_{\pi \in K} \pi G$. Clearly, $\pi G = \langle \pi \rangle G$ and for $H \leqslant K \leqslant S_n$ the inclusion $KG \subseteq HG$ holds.

In [1] W. Kappe proved that

$$R_2G = \{a \in G : [a, x, x] = 1\ \forall x \in G\}$$

is a characteristic subgroup of G and that $[a, x, y] = [a, y, x]^{-1}$ for all $x, y \in G$. Thus $R_2G \subseteq \bar{\pi}G$, where $\pi = (1, 2) \in S_2$. More generally, he showed that for $x, z_i \in G$ the set

$$B_{n-2}(G) = \{a \in G : [a, x, z_1, z_2, \ldots, z_{n-2}, x] = 1\}$$

is a characteristic subgroup of G with the property that

$$[a, x, z_1, z_2, \ldots, z_{n-2}, y] = [a, y, z_1, z_2, \ldots, z_{n-2}, x]^{-1}$$

for all $x, y \in G$. Hence $B_{n-2}(G) \subseteq \bar{\pi}(G)$, where $\pi = (1, n) \in S_n$.

Here we show that πG and $\bar{\pi}G$ are characteristic subgroups of G when 1 and n belong to the same cyclic component of $\pi \in S_n$, $n \geqslant 2$.

2 Notation

Let a, b, c be elements of a group G. Then $[a, b] = a^{-1}b^{-1}ab$, and $a^b = a[a, b] = b^{-1}ab$. The following identities are used.

C(1) (i) $[a, bc] = a^{-1}a^{bc} = a^{-1}a^c a^{-c} a^{bc} = [a, c][a, b]^c$

(ii) $[bc, a] = [b, a]^c[c, a] = [a, b, c]^{-1}[b, a][c, a]$

C(2) $[a^{bc}, [c, b]] = a^{-bc}a^{cb} = [a, b, c]^{-1}\big[[a, b], [a, c]\big][a, c, b]$

C(3) For normal subgroups A, B, C of G, C(2) implies that

$$[A, [B, C]] \leqslant [A, B, C][A, C, B]$$

since $[C,B]=[B,C]$ and $[A,C]\leqslant C$. This is called the *Three Subgroup Lemma*.

$\Gamma_n G$ and $Z_n(G)$ represent the n-th term of the lower and upper central series, respectively.

Remark 2.1 By replacing c by c^{-1} in identity C(2), one obtains the more familiar Hall-Witt identity:

$$[a,b,c^a][c,a,b^c][b,c,a^b]=1.$$

This can also be derived from the identity $[a,b]^c=[a^c,b^c]$ noting that

$$[a^c,b^c]=\left([a,b^c][c,a,b^c]^{-1}\right)^{[a,c]}$$

and hence $[a,b]^{c^a}=[a,b^c][c,a,b^c]^{-1}$, which gives the Hall-Witt identity.

3 Preliminaries

We begin with some elementary results for $n<5$. These will be used for inductive purposes later.

Lemma 3.1 *Let* $\pi=(1,2)\in S_2$*, then* $\pi G=C_G(\Gamma_2 G)$.

Proof — Let $a\in\pi G$: $[a,x,y]=[a,y,x]$ for all $x,y\in G$. Therefore $[a,x,a]=1=[a^x,a]$ for all x in G and hence $N=\langle a^G\rangle$ is abelian. Using C(1), it follows that $a^{yx}=a^{xy}$ and thus $a\in C_G(\Gamma_2 G)$. Conversely, when $a\in C_G(\Gamma_2 G)$, then N is abelian and $a^{xy}=a^{yx}$. Hence by C(2) $[a,x,y]=[a,y,x]$ for all $x,y\in G$. □

Remark 3.2 *Let* $a,b\in G$ *and let* $[b,x,a]=1$ *for all* $x\in G$*. Then* $b\in Z(G/C_G(N))$.

Proof — Since $[b,yx]=[b,x][b,y]^x$, it follows that $[[b,y]^x,a]=1$ for all $x,y\in G$, and hence $[b,y]\in C_G(N)$ for all $y\in G$. □

Lemma 3.3 *Let* $a\in\overline{\pi}G$*, where* $\pi=(1,2)\in S_2$*. Then:*

(1) $N=\langle a^G\rangle$ *is abelian;*

(2) $\overline{\pi}G\subseteq S_3G$*;*

(3) $a^2 \in Z_3G$;

(4) $[ag, x, y] = [a, x, y]^g[g, x, y]$ *for all* $x, y, g \in G$;

(5) $\overline{\pi}G$ *is a characteristic subgroup of* G;

(6) *for* $a \in G$, $a \in \overline{\pi}G$ *if and only if* $[a, [x, y]] = [a, x, y]^2$ *for all* $x, y \in G$.

Proof — (1) Let $a \in \overline{\pi}G$: $[a, x, y] = [a, y, x]^{-1}$ for all $x, y \in G$. Then $[a, x, a] = 1 = [a^x, a]$ for all x and so N is abelian.

(2)–(3) Replacing x by xt, we obtain

$$[a, t, y][a, x, y][a, x, t, y] = [a, y, t]^{-1}[a, y, x]^{-1}[a, y, x, t]^{-1}$$

and hence

$$[a, x, t, y] = [a, y, x, t]^{-1} = [a, t, y, x] = [a, x, t, y]^{-1}.$$

Therefore $[a, x, t, y]^2 = 1$ and $[a, x, t, y] = [a, y, x, t]$. Hence $a \in \sigma G$, where $\sigma = (1, 2, 3) \in S_3$. Also since $[a, x, t, y] = [a, y, t, x]$, it follows that $a \in \tau G$, where $\tau = (1, 3) \in S_3$. Since $\langle \tau, \sigma \rangle = S_3$ and $[a, x, t, y]^2 = 1 = [a^2, x, t, y]$, (2) and (3) follow.

(4) Since

$$[a, x, t, y] = [a, y, x, t] = [a, x, y, t],$$

it follows that $[a, x] \subseteq C_G(\Gamma_2G)$ and

$$[ag, x] = [a, x]^g[g, x] = [g, x][a, x]^g.$$

Therefore

$$[ag, x, y] = [g, x, y][a, x, y]^g = [a, x, y]^g[g, x, y].$$

(5) Follows directly from (4).

(6) Since N is abelian, using C(2) we get

$$[a^{yx}, [x, y]] = [a, [x, y]] = [a, y, x]^{-1}[a, x, y] = [a, x, y]^2.$$

Conversely, when $[a, x, y]^2 = [a, [x, y]]$ for all $x, y \in G$, then

$$[a, [a, y]] = 1 = [a^y, a]$$

and N is abelian. Then by C(2),

$$[a,x,y]^2 = [a,[x,y]] = [a,y,x]^{-1}[a,x,y]$$

and thus $[a,x,y] = [a,y,x]^{-1}$. □

Remark 3.4 The marginal subgroup concept introduced by P. Hall is described in [1]. For example, let

$$w = w(g,x,y) = [g,x,y][g,y,x].$$

The first marginal subgroup of w is defined to be the set

$$\{a \in G : w(ag,x,y) = w(g,x,y)\ \forall g,x,y\}.$$

This set is always a group and the proof of (4), given above, shows that it contains $\bar{\pi}G$. Similarly, it follows from Lemma 3.3 that $\bar{\pi}G$ is also contained in the first marginal subgroup of the word $z = z(g,x,y) = [g,[x,y]][g,x,y]^2$.

The elements a in G for which $[a,[x,y]] = [a,x,y][a,y,x]^{-1}$ for all $x,y \in G$ are similar to the ones featured in a Lie algebra: $(x,y) = xy - yx$. These group elements play a central role in our results. The following special case of identity C(2) is helpful in our deliberations.

Lemma 3.5 *Let* $a,b,c \in G$ *satisfy the following conditions:*

(1) $v = [a,cb]$ *commutes with* $[b,c]$*;*

(2) $[a,[b,c]]$ *commutes with* $z = [a,b][a,c]$.

Then $[a,[b,c]] = \big[[a,c],[a,b]\big][a,b,c][a,c,b]^{-1}$.

Proof — Let

$$[a,cb] = v = [a,b][a,c][a,c,b] = z[a,c,b].$$

Then

$$[a^{cb},[b,c]] = [a,b,c]^v = [a,b,c]^{[a,c,b]}$$

by condition (2). By C(2)

$$[a^{cb},[b,c]] = [a,c,b]^{-1}\big[[a,c],[a,b]\big][a,b,c].$$

Therefore
$$[a,[b,c]] = \big[[a,c],[a,b]\big][a,b,c][a,c,b]^{-1}.$$
The statement is proved. □

Lemma 3.6 *Let* $L(G) = \big\{a \in G : \big[[a,x],[x,y]\big] = 1, \forall x,y \in G\big\}$. *Then the following statements hold.*

(1) $L(G)$ *is a characteristic subgroup of* G.

(2) $H = [a,G]$ *is an abelian normal subgroup of* G *for all* $a \in L(G)$.

(3) $[a,[x,y]] = [a,x,y][a,y,x]^{-1}$ *for all* $x,y \in G$ *and* $a \in L(G)$.

(4) $[a,[x,y,z]] = [a,x,y,z][a,y,x,z]^{-1}[a,z,y,x][a,z,x,y]^{-1}$ *for all* $x,y,z \in G$ *and* $a \in L(G)$.

Proof — (1) Let $a \in L(G)$ and $x,y,z \in G$. Then $[a,x]$ commutes with
$$[x,yz] = [x,z][x,y]^z$$
and hence with $[x,y]^z$. For $z = a$, $[a,x]$ commutes with $[x,y]^a$ and hence $a^{-1} \in L(G)$. Similarly let $b \in L(G)$. Since
$$[ab,x] = [a,x]^b[b,x],$$
it follows that $[ab,x]$ commutes with $[x,y]$ for all $x,y \in G$.

(2) Let $a \in L(G)$ and $x,y \in G$. Since $[a,ax]$ commutes with $[ax,y]$, it also commutes with $[a,y]^x$. Thus H is an abelian normal subgroup of G.

(3) This follows from Lemma 3.5.

(4) Let $a \in L(G)$ and $x,y,z \in G$. Clearly, $[x,y,z] = [c,z]$, where $c = [x,y]$. Therefore
$$[a,c] = [a,x,y][a,y,x]^{-1}$$
and
$$[[a,c],z] = [a,x,y,z][a,y,x,z]^{-1}.$$
Similarly
$$[[a,z],c] = [a,z,x,y][a,z,y,x]^{-1}.$$

Therefore

$$\begin{aligned}[a,[x,y,z]] &= [[a,c],z][[a,z],c]^{-1}\\ &= [a,x,y,z][a,y,x,z]^{-1}[a,z,y,x][a,z,x,y]^{-1}.\end{aligned}$$

The statement is proved. □

Remark 3.7 Let M be a normal subgroup of G such that $[M,G]$ is contained in $C_G(\Gamma_2 G)$, then $M \leqslant L(G)$, since $[[m,x],[x,y]] = 1$ for all $x,y \in G$, $m \in M$. In particular, $S_3G \subseteq L(G)$ and $Z_3G \leqslant L(G)$.

Lemma 3.8 *Let* $\pi = (1,2,3) \in A_3$. *Let* $a \in \pi G$ *and let* $N = \langle a^G \rangle$. *Then:*

(1) $[a,x,y] \in Z(N)$ *for all* $x,y \in G$.

(2) $a \in S_4G$, $a \in L(G)$, $a \in C_G(\Gamma_3 G)$.

(3) $[ag,x,y] = [a,x,y][a,x,g,y][g,x,y]$ *for all* $x,y,g \in G$.

(4) A_3G *is a characteristic subgroup of* G.

Proof — (1) For $a \in \pi G$:

$$[a,x,y,z] = [a,z,x,y] = [a,y,z,x]$$

for all $x,y,z \in G$. For $z = a$, $[a,x,y,a] = 1$ and $[a,x,y] \in Z(N)$ for all $x,y \in G$, by Remark 3.2. This proves (1).

(2) By expanding the identity

$$[a,x,yt,z] = [a,z,x,yt],$$

we obtain the extended symmetry

$$[a,x,y,t,z] = [a,z,x,y,t]$$

for all $x,y,t \in G$. Hence $a \in \tau G$, where $\tau = (1,2,3,4) \in S_4$. Since $\langle \pi,\tau \rangle = S_4$, $a \in S_4G$. Since $S_2G = C_G(\Gamma_2 G)$ it follows from the remark above that $S_3G \subseteq L(G)$. Let $d = [a,x]$. Then $d \in S_3G$ and

$$[d,[x,y]] = [d,x,y][d,y,x]^{-1} = [a,x,x,y][a,x,y,x]^{-1} = 1,$$

since

$$[a,x,x,y] = [a,x,y,x],$$

when $a \in A_3G$. Therefore $\big[[a,x],[x,y]\big] = 1$ for all $x, y \in G$. Since

$$[a,[x,y,z]] = [a,x,y,z][a,y,x,z]^{-1}[a,z,y,x][a,z,x,y]^{-1}$$

by Lemma 3.6, it follows that $[a,[x,y,z]]=1$, since $[a,x,y,z]=[a,z,x,y]$ and $[a,y,x,z]=[a,z,y,x]$ for all $x,y,z \in G$. Therefore $a \in C_G(\Gamma_3 G)$. Thus (2) is established.

(3) $[ag,x] = [a,x][a,x,g][g,x]$ and $[ag,x,y] = [a,x,y][a,x,g,y][g,x,y]$, since $[a,x,y] \in Z(N)$ and $[[a,x],[y,x]] = 1$, since $a \in L(G)$.

(4) Follows from (3). □

Lemma 3.9 *Let* $\pi=(1,4)(2,3) \in S_4$, $a \in \pi G$ *and* $N = \langle a^G \rangle$. *Then* $\pi G \subseteq S_5 G$.

Proof — For $a \in \pi G$, $[a,x,y,z,w] = [a,w,z,y,x]$ for all $x,y,z,w \in G$. In particular, $[a,x,y,z,a] = 1$ and $[a,x,y,z] \in C_G(N)$ for all $x,y,z \in G$ by Remark 3.2. By expanding the defining relation, we get the extended relation

$$[a,x,y,t,z,w] = [a,w,z,y,t,x]$$

on replacing y by yt. Therefore $a \in \gamma G$, where $\gamma = (1,5)(2,3,4) \in S_5$. Since $\langle \pi, \gamma \rangle = S_5$ the result follows. □

4 Permutation extension and commutator expansion

Let $\pi \in S_n$ and let

$$\pi : (x_1, x_2, \dots, x_n) \longrightarrow (y_1, y_2, \dots, y_n).$$

Let $\pi : x_r \mapsto y_k$. We view π as a positional permutation and write $\pi : r \mapsto k$. If a new element $t \in G$ is placed immediately after x_r on both sides, we obtain the extended permutation:

$$\pi_r : (x_1, \dots, x_r, t, x_{r+1}, \dots, x_n) \to (y_1, \dots, y_{k-1}, x_r, t, y_{k+1}, \dots, y_n).$$

Using the natural embedding of π into S_{n+1}, we obtain a subgroup $E(\pi)$ of S_{n+1}, where

$$E(\pi) = \langle \pi, \pi_1, \dots, \pi_n \rangle.$$

$E(\pi)$ is the extension group defined by π.

Example Let $\pi = (1,2)(4,5)(3) \in S_5$. Then $\pi_1 = (1,2,3)(5,6) = \pi_2^{-1}$, $\pi_3 = (1,2)(5,6)(4)$, $\pi_4 = (1,2)(4,5,6)(3) = \pi_5^{-1}$ and thus $E(\pi)$ is the direct product of 2 copies of S_3.

We require some further notation on commutator expansions. Let N be a normal subgroup of G. The series of normal subgroups defined inductively by: $N_0 = N$, $N_{i+1} = [N_i, G]$ for $i > 0$ is a decreasing lower central G-series in N. Let M and N be normal subgroups of G and let $K = K^{(0)} = [M, N]$. For $r > 0$,

$$K^{(r)} = \prod_{i+j=r} [M_i, N_j].$$

The series defined by $K^{(r)}$ is also a decreasing lower central G-series in G, since

$$[M_i, N_j, G] \leqslant [M_{i+1}, N_i][M_i, N_{j+1}]$$

by the Three Subgroup Lemma. Thus $K^{(r+1)} \leqslant K^{(r)}$.

Lemma 4.1 *Let* $R_r = [M, N_r][M_r, N]$, $r > 3$. *Then* $K^{(r)} = R_r C^{(r-2)}$, *where* $C = [M_1, N_1]$.

Proof — Let $M_1 = A$ and $N_1 = B$, then $C = [A, B]$. Let $i + j = r$, $i, j > 0$. Then

$$[M_i, N_j] = [A_{i-1}, B_{j-1}] \leqslant C^{(r-2)}$$

and so $K^{(r)} = R_r C^{(r-2)}$. □

Lemma 4.2 *Let* N *be a normal subgroup of* G *and let* $K = [N, N]$. *When* $[N, N_{r+1}] = 1$ *and* $K^{(r)} \leqslant Z(G)$, *then* $K^{(r+1)} = 1$.

Proof — Since $[N, N_{r+1}] = 1$, the leading term of the product defining $K^{(r+1)}$ is $[N_1, N_r]$. K^r has leading term $[N, N_r]$. Let $[a, b]$ in $[N, N_r] \leqslant Z(G)$. Then $[a, b, g] = 1$ for all $g \in G$. Since $\big[[g, a], [g, b]\big]$ belongs to $[N_r, N_{r+1}] = 1$, using $C(2)$, it follows that $[g, a, b] = 1$. Thus $[g, b, a] \in [N_{r+1}, N] = 1$ for all $g, b, a \in G$ and so $[N_1, N_r] = 1$.

Inductively, using this argument we conclude that every term in $K^{(r+1)}$ is trivial. □

Theorem 4.3 *Let* $[N, N_n] = 1$ *and let* $K^{(n)} = K^{(n+1)}$, *then* $K^{(n)} = 1$, *where* $K = [N, N]$.

Proof — Suppose not. There exists an integer $0 < r < n$, such that $K^{(r)} = K^{(r+1)} \neq K^{(r+2)}$. Since

$$[K^{(r)}, G] = [K^{(r+1)}, G] \leqslant K^{(r+2)},$$

it follows that $K^{(r)} \leqslant Z(G) \bmod K^{(r+2)}$ and hence $K^{(r+1)} \leqslant K^{r+2}$, by Lemma 4.2. So $K^{(r)} = K^{(r+1)} = K^{(r+2)}$, a contradiction. □

We now establish some results on commutator extensions.

Lemma 4.4 *Let* M *be a normal subgroup of* G. *Let* $u, v \in M$. *Let*

$$u_k = [u, x_1, x_2, \dots, x_k] \quad \text{and} \quad v_k = [v, x_1, x_2, \dots, x_k],$$

where $x_i \in G$ *and* $k \geqslant 1$. *Then* $[uv, x_1, x_2, \dots, x_k] \equiv u_k v_k$ *mod* $B^{(k)}$ *for all* k, *where* $B = [M, M]$.

Proof — For $k = 1$

$$[uv, x_1] = [u, x_1]^v [v, x_1] = u_1 v_1 \mod [M_1, M] = B^{(1)}.$$

Assume the result is true for $r = k - 1$ Then

$$[uv, x_1, x_2, \dots, x_r] = abc,$$

where $a = u_r$, $b = v_r$ and $c \in B^{(r)}$. Let $d = x_{r+1}$. Then

$$\begin{aligned}[abc, d] = [a, d]^{bc}[b, d]^c[c, d] &\equiv [a, d][b, d] \mod B^{(r+1)} \\ &\equiv u_{r+1} v_{r+1} \mod B^{(r+1)}\end{aligned}$$

and the result follows by induction. □

Corollary 4.5 *Let* $a \in G$ *and let* $H = [a, G] = \langle [a, g], \forall g \in G \rangle$. *Let* $C = [H, H]$ *and* $x_1, x_2, \dots, x_n, t \in G$. *Then*

$$[a, x_1 t, x_2, \dots, x_n] \equiv u_n v_n w_n \mod C^{(n-1)},$$

where

$$u_n = [a, t, x_2, \dots, x_n], \quad v_n = [a, x_1, x_2, \dots, x_n]$$
$$\textit{and} \quad w_n = [a, x_1, t, x_2, \dots, x_n].$$

Proof — Since $[a, x_1 t] = u_1 v_1 w_1$, where $u_1 = [a, t]$, $v_1 = [a, x_1]$, $w_1 = [a, x_1, t]$, the result follows from Lemma 4.4, by replacing u by u_1 and v by $v_1 w_1$ and replacing M by H. □

Theorem 4.6 *Let* $a \in G$, $H = [a, G]$, $C = [H, H]$ *and* $C^{(n-1)} = 1$. *If* $a \in \pi G$, *then* $a \in E(\pi)G$.

Proof — Let $b = [a, z_1, z_2, \ldots, z_r] \in H_{r-1}$, $z_i \in G$. Let $[b, G] = B$. Then $[B, B]^{(n-r)} \leqslant C^{(n+r-2)} = 1$ for $r \geqslant 1$. Let π map $(x_1, x_2, \ldots, x_n)$ to $(y_1, y_2, \ldots, y_n)$. Let $x_1\pi = y_j$, where $j \leqslant i$. Then

$$[a, x_1, x_2, \ldots, x_n] = [a, y_1, y_2, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_n]$$

and hence $a \in \pi_1 G$. Replacing x_1 by $x_1 t$ on both sides we obtain

$$[a, x_1 t, x_2, \ldots, x_n] = u_n v_n w_n,$$

by Corollary 4.5, where $u_n = [a, t, x_1, x_2, \ldots, x_n]$, $v_n = [a, x_1, x_2, \ldots, x_n]$ and $w_n = [a, x_1, t, x_2, \ldots, x_n]$.

Similarly, the right-hand side becomes $\overline{u}_n \overline{v}_n \overline{w}_n$, where

$$\overline{u}_n = [a, y_1, y_2, \ldots, y_{j-1}, t, x_1, y_{j+1}, \ldots, y_n],$$
$$\overline{v}_n = [a, y_1, y_2, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_n] \quad \text{and}$$
$$\overline{w}_n = [a, y_1, y_2, \ldots, y_{j-1}, x_1, t, y_{j+1}, \ldots, y_n].$$

Since $u_n = \overline{u}_n$, $v_n = \overline{v}_n$ for $a \in \pi G$, it follows that

$$[a, x_1, t, x_2, \ldots, x_n] = [a, y_1, y_2, \ldots, y_{j-1}, x_1, t, y_{j+1}, \ldots, y_n]$$

and hence $a \in \pi_1 G$.

This argument shows that when $x_r\pi = y_k$, $r < k$, then $a \in \pi_r G$. When $k < r$, then $a \in \sigma_k G$, where $\sigma = \pi^{-1}$ and $\sigma_k = \pi_r^{-1}$. □

5 Properties of $A_{n+1}(G)$

We have noted earlier that the commutator $[a, [b, c]]$ is a product of eight simple commutators. For $a \in L(G)$ this reduces to the product of $[a, b, c]$ and $[a, c, b]^{-1}$, $a, b, c \in G$.

Let $x_1, x_2, \ldots, x_n$ be a sequence of elements belonging to a group G. Let $a \in G$. We make the following definitions:

$$a\delta(x_1, x_2) = [a, x_1, x_2][a, x_2, x_1]^{-1},$$

for $2 \leqslant i \leqslant n$,

$$a\delta_{i+1}(x_1, x_2, \ldots, x_{i+1}) \\ = [a\delta(x_1, x_2, \ldots, x_i), x_{i+1}]\{[a, x_{i+1}]\delta(x_1, \ldots, x_i)\}^{-1}.$$

For example:

$$\begin{aligned} a\delta_3(x, y, z) &= [a\delta_2(x, y), z]\{[a, z]\delta(x, y)\}^{-1} \\ &= \big[[a, x, y][a, y, x]^{-1}, z\big]\{[a, z, x, y][a, z, y, x]\}^{-1} \\ &\equiv [a, x, y, z][a, y, x, z]^{-1}[a, z, y, x][a, z, x, y]^{-1} \quad \text{mod } [H_2, H_1] \leqslant C^{(3)}, \end{aligned}$$

where $[a, G] = H$ and $C = [H, H]$. Thus, if $a \in A_4(G)$ and $C^{(3)} = 1$, then

$$[a, [x, y, z]] = a\delta(x, y, z)$$

is a product of 4 simple commutators of alternating sign. The result in this case is a consequence of the following theorem.

Theorem 5.1 *Let* $a \in A_{n+1}(G)$, $n \geqslant 2$. *Let* $N = \langle a^G \rangle$ *and* $K = [N, N]$. *Then:*

(1) $K^{(n)} = 1$.

(2) $[N, \Gamma_{n+1}(G)] = 1$.

(3) $[a, [x_1, x_2, \ldots, x_n]] = a\delta(x_1, x_2, \ldots, x_n)$.

(4) $[ag, x_1, x_2, \ldots, x_n] =$
$= [a, x_1, x_2, \ldots, x_n][a, x_1, g, x_2, \ldots, x_n][g, x_1, x_2, \ldots, x_n]$
for all $x_i, g \in G$.

Proof — For $n = 2$ this result is contained in Lemma 3.8. We can assume that (i) $aZ \in A_n(G/Z(G))$ and (ii) $[a, g] \in A_n(G)$ for all $g \in G$.

(1) Since the permutation $(1, n+1)(2, n) \in A_{n+1}$, it follows that

$$[a, x_1, x_2, \ldots, x_n, a] = 1$$

and thus $[N, N_n] = 1$. Hence we can assume that $K^{(n-1)} \leqslant Z(G)$. So for $r = n - 1$, $[N, N_{r+1}] = 1$ and $K^{(n-1)} \leqslant Z(G)$, and therefore $K^{(r+1)} = K^{(n)} = 1$ by Lemma 4.2.

(2) By (i) and (ii) we can further assume that $[N, \Gamma_n(G)] \leqslant Z(G)$ and $[N_1, \Gamma_n(G)] = 1$. Therefore

$$[\Gamma_{n+1}(G), N] = [[\Gamma_n(G), G], N] \leqslant [[N, \Gamma_n(G)], G][N_1, \Gamma_n(G)] = 1$$

(3) Let $b = [x_1, x_2, \ldots, x_{n-1}]$ and $c = x_n$. Then

$$[x_1, x_2, \ldots, x_n] = [b, c] \in \Gamma_n(G)$$

and

$$[a, [x_1, x_2, \ldots, x_n]] = [a, [b, c]] \in \Gamma_{n+1}(G).$$

Using (ii) we see that a, b, c satisfy the conditions of Lemma 3.5 and so

$$[a, [b, c]] = [a, b, c][a, c, b]^{-1}.$$

By (i),

$$[a, b] = [a, [x_1, x_2, \ldots, x_{n-1}]] \equiv a\delta(x_1, x_2, \ldots, x_{n-1}) \mod Z(G)$$

and hence $[a, b, c] = [a\delta(x_1, x_2, \ldots, x_{n-1}), x_n] = 1$, since $K^{(n)} = 1$. By (ii),

$$[a, c, b] = [a, x_n]\delta(x_1, x_2, \ldots, x_{n-1}).$$

Therefore

$$\begin{aligned} &[a, [x_1, x_2, \ldots, x_n]] \\ &= [a\delta(x_1, x_2, \ldots, x_{n-1}), x_n]\{[a, x_n]\delta(x_1, x_2, \ldots, x_{n-1})\}^{-1} \\ &= a\delta(x_1, x_2, \ldots, x_n). \end{aligned}$$

This proves (3).

(4) By (i), we may assume that

$$[ag, x_1, x_2, \ldots, x_{n-1}] \equiv uvw \mod Z(G),$$

where $u = [a, x_1, x_2, \ldots, x_{n-1}]$, $v = [a, x_1, g, x_2, \ldots, x_{n-1}]$ and $w = [g, x_1, x_2, \ldots, x_{n-1}]$. Since $[a, x_1, x_2, \ldots, x_{n-2}] \in A_3(G) \leqslant C_G(\Gamma_3(G))$, it follows that

$$[ag, x_1, x_2, \ldots, x_n] = [u, x_n][v, x_n][w, x_n]$$

and the result is established. $\square$

6 Properties of $E(\pi)G$ for $\pi \in S_n$

In this section we consider properties of $E(\pi)G$, for $\pi \in S_n$. In particular, the permutations π for which 1 and n are co-cyclic in π. For every $\pi \in S_n$, the group $E(\pi)$ is defined as follows. Let

$$\pi : (x_1, x_2, \ldots, x_n) \longrightarrow (y_1, y_2, \ldots, y_n)$$

and let $x_r\pi = y_k$. We will describe this as $r\pi = k$ and $r = k\sigma$ where $\sigma = \pi^{-1}$. We describe $E(\pi)$ using this notation. Let $1 \leqslant r \leqslant n$ and define π_r in the following way:

$$r\pi_r = k,$$
$$(r+1)\pi_r = k+1,$$

for $1 \leqslant i \leqslant r-1$,

$$i\pi_r = \begin{cases} i\pi & \text{if } i\pi < k \\ i\pi + 1 & \text{if } i\pi > k \end{cases},$$

for $r+1 \leqslant j \leqslant n$,

$$(j+1)\pi_r = \begin{cases} j\pi & \text{if } j\pi < k \\ j\pi + 1 & \text{if } j\pi > k \end{cases}$$

Here $\sigma_k = \pi_r^{-1}$, where $\sigma = \pi^{-1}$. As outlined earlier

$$E(\pi) = \langle \pi, \pi_1, \ldots, \pi_n \rangle \leqslant S_{n+1}.$$

For example, let $1\pi = n$. Then $1\pi_1 = n$, $2\pi_1 = n+1$ and for $2 \leqslant j < n$,

$$(j+1)\pi_1 = j\pi, \qquad (n+1)\pi_1 = n\pi,$$

$\alpha_1 = \pi_1\pi^{-1} = (1)\ (n+1, n, \ldots, 3, 2)$ and $\langle \pi, \pi_1 \rangle = S_{n+1}$.

Let $\pi = \gamma\rho$, where $\gamma = (1, a_1, \ldots, a_k)$, and $1\pi = a_1$, $a_i\pi = a_{i+1}$ for $2 \leqslant i \leqslant k-1$, $a_k\pi = 1$. In Theorem 6.5 we show that $A_{n+1}(G) \leqslant E(\pi)$, when 1 and n are co-cyclic in π and so belong to the set $\{1, a_1, \ldots, a_k\}$. Throughout this section, $r\pi = n = s\sigma$, $\sigma = \pi^{-1}$. By the example given above, we can assume $1 < r,\ s < n$ and hence $k \geqslant 4$.

Lemma 6.1 *Let* $\alpha_r = \pi_r\pi^{-1}$ *and* $\beta_s = \sigma_s\sigma^{-1}$. *Let* $S = \mathrm{Stab}(n+1) \leqslant S_{n+1}$. *Then:*

(1) α_r *and* β_s *are cycles in* S_{n+1}*;*

(2) $(n+1, n, s) = [\alpha_r, \beta_s]$, *where* $r < s$*;*

(3) $(n, r, s) \in S$.

PROOF — $r\pi_r = r\pi = n$, $(r+1)\pi_r = n+1$. For $1 \leqslant i \leqslant r-1$, $i\pi_r = i\pi$, $i\alpha_r = i$. For $r+2 \leqslant j \leqslant n$, $(j+1)\pi_r = j\pi$, $(j+1)\alpha_r = j$. Thus

$$\alpha_r = (n+1, n, \dots, r+2, r+1).$$

Similarly $\beta_s = (n+1, n, \dots, s+2, s+1)$. When $r < s$ the commutator $[\alpha_r, \beta_s] = (n+1, n, s)$. When $r > s$, $[\alpha_r, \beta_s] = (n+1, r, n)$. Since $(n+1, n, s)^\sigma = (n+1, r, n)$, it follows that both belong to S_{n+1} and their product $(n, s, r) \in \mathrm{Stab}(n+1) = S$. □

Lemma 6.2 *Let* x, y, z *be a triple of non-zero integers* $\leqslant n$. *Let* $x\pi = z$ *and* $z\pi = y$. *Let* $x, y < z$. *Then:*

(1) $(z+1)\pi_x\sigma_y = z$*;*

(2) *when* $z < s$, $\pi_x\pi_y \in \mathrm{Stab}(n+1) \leqslant S_{n+1}$.

PROOF — (1) $(z+1)\pi_x = z\pi = y$, since $z\pi = y < x\pi = z$. Moreover, $y\sigma_y = y\sigma = z$ and so $(z+1)\pi_x\sigma_y = z$.

(2) When $z < s = n\pi$, $(n+1)\pi_x = n\pi + 1 = s+1$, and $(s+1)\sigma_y = s\sigma + 1 = n+1$. So $\pi_x\sigma_y \in \mathrm{Stab}(n+1)$. □

Lemma 6.3 *Let* z *be a fixed point of* π. *Let* a, b *be positive integers such that* $a < z < b < n$, $a\pi = b$. *Then:*

(1) $(z+1)\sigma_z\pi_a = z$

(2) *when* $z < r$, *the permutation* $\sigma_z\pi_a \in \mathrm{Stab}(n+1)$.

PROOF — (1) $(z+1)\sigma_z = z\sigma + 1 = z+1$ and $(z+1)\pi_a = z\pi = z$ since $a < z < a\pi = b$. Therefore $(z+1)\sigma_z\pi_a = z$.

(2) When $z < r$, $(n+1)\sigma_z = n\sigma + 1 = r+1$, since $r > z\sigma = z$. Also $(r+1)\pi_a = r\pi + 1 = n+1$, since $r\pi = n > a\pi$. Therefore $\sigma_z\pi_a$ belongs to $\mathrm{Stab}(n+1)$. □

Lemma 6.4 $E(\pi)$ *is a transitive subgroup of* S_{n+1}, *when* 1 *and* n *are co-cyclic in* π.

Proof — Suppose not. Then there exists a positive integer z, such that $\{n+1, n, \ldots, z+1\}$ belong to the orbit of $n+1$, but no element of $E(\pi)$ maps $z+1$ to z. We consider two cases.

(1) z is *not* a fixed point of π.

Then it follows, by Lemma 6.1, that $z < r$, $z < s$, since $\alpha_r, \beta_s \in S_{n+1}$. So $z\pi = y$ and $x = z\sigma$ have the property that $\pi_x \sigma_y$ maps $z+1$ to z, by Lemma 6.2.

(2) z *is* a fixed point of π.

Let γ be the cycle containing 1 and n. Then

$$z \notin \{1, a_1, \ldots, a_k\} \quad \text{and } 1 < z < n.$$

There exists a largest integer a_i such that $a_i < z$ and $z < a_{i+1}$. Let $a = a_i$, then $a\pi = b = a_{i+1}$. We can assume $a < b$, since if $a > b$ then $z\sigma = z$ and $b\sigma = a > b$. Then, by Lemma 6.3, $\sigma_z \pi_a$ maps $z+1$ to z. Therefore $E(\pi)$ is transitive. □

Theorem 6.5 *Let 1 and n belong to the same cycle in π. Then $A_{n+1} \leqslant E(\pi)$.*

Proof — $E(\pi)$ is a transitive group containing three cycles

$$(n+1, n, s),\ (n+1, r, n),\ (n, s, r), \quad n \geqslant 5.$$

By a well-known result of Marggraf, it suffices to show that $E(\pi)$ is 2-transitive. Thus it suffices to show that $S = \mathrm{Stab}(n+1)$ acts transitively on the set $\{1, 2, \ldots, n\}$. Suppose not. Then there exists z in $\{1, 2, \ldots, n\}$ with the property that no element of S maps $z+1$ to z. Then, using Lemma 6.2 (2) and Lemma 6.3 (2), we obtain the required contradiction. □

Theorem 6.6 *Let 1 and n be co-cyclic in $\pi \in S_n$. Let $a \in \pi G$, then $a \in E(\pi)G$.*

Proof — Let $\langle a^G \rangle = N$ and $K = [N, N]$. Let $H = [a, G]$ and $C = [H, H]$. Then, by Theorem 4.6, it suffices to show that $C^{(n-1)} = 1$. Suppose not. Then by Theorem 6.5, $E(\pi)G \leqslant A_{n+1}G \bmod C^{(n-1)}$. Therefore, by Theorem 5.1, $K^{(n)} \equiv 1 \bmod C^{(n-1)}$ and thus

$$K^{(n)} \leqslant C^{(n-1)} \leqslant K^{(n+1)}.$$

Hence $K^{(n)} = K^{(n+1)}$ and, by Theorem 4.3, $K^{(n)} = 1 = C^{(n-1)}$. □

Theorem 6.7 *Let* 1 *and* n *be co-cyclic in* $\pi \in S_n$. *Then:*

(1) πG *is contained in the first marginal subgroup of the group word* $w \equiv [x_1, x_2, \ldots, x_n][y_1, y_2, \ldots, y_n]^{-1}$ *for* $x_i, y_i \in G$.

(2) $\overline{\pi}G$ *is contained in the first marginal subgroup of*

$$\overline{w} \equiv [x_1, x_2, \ldots, x_n][y_1, y_2, \ldots, y_n].$$

Proof — (1) We are required to show that

$$\begin{aligned}&[ag, x_1, x_2, \ldots, x_n][ag, y_1, y_2, \ldots, y_n]^{-1}\\ &= [g, x_1, x_2, \ldots, x_n][g, y_1, y_2, \ldots, y_n]^{-1}\end{aligned}$$

for $g \in G$. Using Theorem 5.1 (4), it suffices to show that

$$[a, x_1, g, x_2, \ldots, x_n] = [a, y_1, g, y_2, \ldots, y_n].$$

When π is an odd permutation, $E(\pi) = S_{n+1}$ and the result is immediate for $a \in E(\pi)G$. When π is an even permutation, then the permutation $\mu \in A_{n+1}$, where $\mu = \tau^{-1}\pi\tau$ and $\tau = (2, 3, \ldots, n+1)$. In sequence μ maps:

$$\begin{aligned}&(x_1, g, x_2, \ldots, x_n) \to (x_1, x_2, \ldots, x_n, g)\\ &\to (y_1, y_2, \ldots, y_n, g) \to (y_1, g, y_2, \ldots, y_n).\end{aligned}$$

Since $a \in \mu G$, it follows that

$$[a, x_1, g_2, x_2, \ldots, x_n] = [a, y_1, g, y_2, \ldots, y_n]$$

and thus (1) is established.

(2) When $a \in \overline{\pi}(G)$, $[a, x_1, x_2, \ldots, x_n] = [a, y_1, y_2, \ldots, y_n]^{-1}$, where

$$\pi : (x_1, x_2, \ldots, x_n) \to (y_1, y_2, \ldots, y_n).$$

In this case $[a, x_1 g, x_2, \ldots, x_n] = [a, y_1, g, y_2, \ldots, y_n]^{-1}$. □

Theorem 6.8 *Let* $\rho : (x_1, x_2, \ldots, x_n) \to (x_n, x_{n-1}, \ldots, x_1)$, $n > 1$. *Then* $\rho G \leqslant C_G(\Gamma_n G)$, *when* n *is even, and* $\overline{\rho}G \leqslant C_G(\Gamma_n G)$, *when* n *is odd.*

Proof — ρ is a product of involutions beginning with $(1, n)$. Let $a \in \rho G$. Then, by Theorem 5.1,

$$[a, [x_1, x_2, \dots, x_n]] = a\delta(x_1, x_2, \dots, x_n).$$

In the expansion of $a\delta(x_1, x_2, \dots, x_n)$, the simple commutators

$$[a, z_1, z_2, \dots, z_n] \quad \text{and} \quad [a, z_n, z_{n-1}, \dots, z_1]$$

occur as a pair, where $\{z_1, z_2, \dots, z_n\}$ is the set $\{x_1, x_2, \dots, x_n\}$. When n is even, these elements have opposite sign and $a\delta(x_1, x_2, \dots, x_n) = 1$. So $[a, [x_1, x_2, \dots, x_n]] = 1$. When n is odd, these elements have the same sign and thus for $a \in \overline{\rho}G$ the expansion of $a\delta(x_1, x_2, \dots, x_n)$ is again equal to 1. □

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