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Groups whose Proper Subgroups of Infinite Rank are Černikov-by-Hypercentral or Hypercentral-by-Černikov *

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Abstract

It is proved that if G is an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral (respectively, hypercentral-by-Černikov), then all proper subgroups of G are Černikov-by-hypercentral (respectively, hypercentral-by-Černikov), where $\mathfrak{X}$ is the class defined by N.S. Černikov as the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, $\mathbf{R}$ and $\mathbf{L}$.

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1 Introduction

A group G is said to have finite rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such a property. If there is no such r, the group G has infinite rank. In recent years, many authors studied the structure of the locally (soluble-by-finite) groups G of infinite rank in which every proper subgroup of infinite rank belongs to a given class $\mathfrak{Y}$ and they proved that all proper subgroups of G belong to $\mathfrak{Y}$,

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sometimes the group G itself belongs to $\mathfrak{Y}$ (see for instance, [1],[3],[5], [6],[7],[8] and [15]). In particular, it is proved in [5, Theorem B′], that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are locally nilpotent is itself locally nilpotent, where $\mathfrak{X}$ is the class introduced in [2] as the class obtained by taking the closure of the class of periodic locally graded groups by the closure operations $\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, **R** and **L** (see [13, Chapter 1] for the definitions of the closure operations). Clearly $\mathfrak{X}$ is a subclass of the class of locally graded groups that contains all locally (soluble-by-finite) groups. Recall that a group is said to be locally graded if every non-trivial finitely generated subgroup contains a proper subgroup of finite index. In [2], it is proved that an $\mathfrak{X}$-group of finite rank is almost locally soluble. Using [5, Theorem B′] and the fact that locally nilpotent groups of finite rank are hypercentral [13, Corollary 1 of Theorem 6.36], one can see that an $\mathfrak{X}$-group of infinite rank whose proper subgroups of infinite rank are hypercentral has all its proper subgroups hypercentral. In the present note, we will consider this problem for the classes of Černikov-by-hypercentral and hypercentral-by-Černikov groups. More precisely, we will prove that an $\mathfrak{X}$-group G of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral (respectively, hypercentral-by-Černikov), has all its proper subgroups Černikov-by-hypercentral (Theorem 9, below) (respectively, hypercentral-by-Černikov (Theorem 11, below)).

Our notation and terminology are standard and follow [13].

2 The Černikov-by-hypercentral case

In this section we consider $\mathfrak{X}$-groups of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral. First let's give a couple of lemmas on Černikov-by-hypercentral groups which are probably known.

Lemma 1 *If* G *is a Černikov-by-hypercentral group, then* G *has a characteristic Černikov subgroup* N *such that* G/N *is hypercentral.*

Proof — Let C be a normal Černikov subgroup of G such that G/C is hypercentral. Consider the non-empty set,

$$\mathcal{H} := \{E \mid E \leqslant C,\ E \lhd G \text{ and } G/E \text{ is hypercentral}\}.$$

Since C satisfies the minimal condition on subgroups, the set $\mathcal{H}$ has a minimal element, say N. Let $\alpha \in \mathrm{Aut}(G)$; clearly $\alpha(N)$ is normal in G and $G/\alpha(N) \simeq G/N$. Since G/N and $G/\alpha(N)$ are hypercentral, so is $G/(N \cap \alpha(N))$. We deduce that $N = N \cap \alpha(N)$ and so $N \leqslant \alpha(N)$. Hence N is characteristic in G. □

Lemma 1 has the following consequence.

Corollary 2 *The class of Černikov-by-hypercentral groups is* $\mathbf{N}_0$*-closed.*

Proof — Let H and K be normal Černikov-by-hypercentral subgroups of a group G. By Lemma 1, there exist two Černikov subgroups A and B which are respectively characteristic in H and K such that H/A and K/B are hypercentral groups. Clearly AB is a normal Černikov subgroup of G and HK/AB is a hypercentral group, as it is the product of its normal hypercentral subgroups HB/AB and KA/AB. Hence HK is a Černikov-by-hypercentral group, as required. □

Lemma 3 *Let* G *be a group without proper subgroups of finite index. If* N *is a normal Černikov-by-hypercentral subgroup of* G*, then* N *is hypercentral.*

Proof — Since N is Černikov-by-hypercentral, by Lemma 1 it has a Černikov characteristic subgroup K such that N/K is hypercentral. Let D be the divisible part of K. Since any abelian Černikov group is covered by finite characteristic subgroups, D is central in G as G has no proper subgroup of finite index. Now K/D is a finite normal subgroup of G/D, so it is central in G/D. Hence K is contained in the second centre of G. It follows that N is hypercentral. □

Lemma 4 *If* G *is a periodic Černikov-by-hypercentral group, then it is hypercentral-by-Černikov.*

Proof — Let N be a normal Černikov subgroup of G such that G/N is hypercentral. Since G is periodic, $G/C_G(N)$ is Černikov [13, Theorem 3.29]. Since G/N is hypercentral, so is $C_G(N)/N \cap C_G(N)$. But $N \cap C_G(N)$ is contained in the centre of $C_G(N)$, hence $C_G(N)$ is hypercentral. Therefore G is hypercentral-by-Černikov. □

Lemma 5 *Let* G *be a locally (soluble-by-finite) group of infinite rank. If all proper subgroups of infinite rank of* G *are Černikov-by-hypercentral, then* G *is either Černikov-by-hypercentral or locally finite.*

Proof — Assume for a contradiction that G is neither locally finite nor Černikov-by-hypercentral. Since proper subgroups of G of infinite rank are (locally finite)-by-(locally nilpotent), so is G by [8, Theorem B]. Let T be the torsion part of G, so T is locally finite and G/T is locally nilpotent and torsion-free. It follows that proper subgroups of G/T are hypercentral by [13, Corollary 1 of Theorem 6.36]. By [9, Theorem 3.8], G/T is hypercentral and so is not perfect. It is easy to see that every proper normal subgroup of G is Černikov-by-hypercentral (for instance by [8, Lemma 2]), so G/G' cannot be the product of two proper subgroups by Corollary 2. It follows that G/G' is periodic and hence so is G/T, a contradiction. Therefore G is as claimed. □

Lemma 6 *Let G be a locally finite group of finite rank with a proper subgroup of finite index. If all proper subgroups of G are Černikov-by-hypercentral, then so is G.*

Proof — By assumption, $G = FN$ where N is a proper normal subgroup of G and F is a finite subgroup of G. As N is Černikov-by-hypercentral, by Lemma 1 we may suppose, without loss of generality, that N is hypercentral. Hence N is the direct product of its Sylow p-subgroups, namely

$$N = \mathop{\mathrm{Dr}}_{p\in\pi(N)} N_p,$$

where $\pi(N)$ is the set of primes dividing the orders of the elements of N. Since locally finite p-groups of finite rank are Černikov [13, Corollary 2 of Theorem 6.36] and since the direct product of finitely many Černikov groups is Černikov, we may assume that N is a direct product of infinitely many Sylow p-subgroups for different primes p. Now we can choose $q \in \pi(N)$ such that $q \notin \pi(F)$. So $N = N_q \times A$, where

$$A := \mathop{\mathrm{Dr}}_{q\neq p\in\pi(N)} N_p.$$

Clearly $B := FA$ is a proper subgroup of G and hence B is Černikov-by-hypercentral. Therefore $G = N_qB$ is Černikov-by-hypercentral and the result follows. □

Lemma 7 *Let G be a locally finite group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral. If G has a proper subgroup of finite index, then all proper subgroups of G are Černikov-by-hypercentral.*

Proof — Assume for a contradiction that the statement is false. Then G contains a proper subgroup H of finite rank which is not Černikov-by-hypercentral. Let N be a proper normal subgroup of finite index in G. Since N is Černikov-by-hypercentral, we may assume by Lemma 1, without loss of generality, that N is hypercentral. Hence

$$N = \underset{p \in \pi(N)}{\mathrm{Dr}} N_p$$

is the direct product of its Sylow p-subgroups. Let p be a prime in $\pi(N)$ such that N_p has infinite rank. Clearly H cannot be contained in any proper subgroup of infinite rank, so $G = HN_p$. Thus G/N_p has finite rank. Since all proper subgroups of G/N_p are Černikov-by-hypercentral, we deduce, by Lemma 6, that G/N_p is Černikov-by-hypercentral. It follows that $H/H \cap N_p$ is Černikov-by-hypercentral. Since $H \cap N_p$ is a locally finite p-group of finite rank, it is Černikov [13, Corollary 2 of Theorem 6.36]. Therefore H is Černikov-by-hypercentral, and this contradiction shows that for all primes p in $\pi(N)$, the subgroups N_p have finite but unbounded ranks. It follows that N has a subgroup $C = N_{p_1} \times N_{p_2} \times \dots$ which is a direct product of infinitely many Sylow p_i-subgroups such that p_i's are distinct primes, the rank $r(N_{p_i})$ of each N_{p_i} is finite and $r(N_{p_i}) < r(N_{p_{i+1}})$ for all positive integer i. Put $C = T \times R$, where $T = N_{p_1} \times N_{p_3} \times \dots$ and $R = N_{p_2} \times N_{p_4} \times \dots$, so T and R are proper normal subgroups of infinite rank of G. It follows that G/T have infinite rank and hence HT is a proper subgroup of infinite rank of G containing H, a contradiction that completes the proof. □

Lemma 8 *If* G *is a locally finite group of infinite rank whose proper subgroups of infinite rank are Černikov-by-hypercentral, then* G *is not simple.*

Proof — Assume for a contradiction that G is simple. Since periodic Černikov-by-hypercentral groups are hypercentral-by-Černikov by Lemma 4, they are almost hypercentral-by-(abelian and periodic) and hence almost locally soluble. Since locally finite groups of finite rank are almost locally soluble [14], we deduce that every proper subgroup of G is almost locally soluble. It follows, by [11], that G is isomorphic to either $PSL(2, F)$ or $Sz(F)$ for some infinite locally finite field F. But each of these groups has a proper non-(hypercentral-by-Černikov) subgroup of infinite rank (see the proof of the Theorem of [12]), hence a proper non-(Černikov-by-hypercentral) subgroup of infinite rank by Lemma 4. This leads to a contradiction. □

Now we are ready to prove the main result of this section.

Theorem 9 *Let* G *be an* $\mathfrak{X}$*-group of infinite rank. If all proper subgroups of infinite rank of* G *are Černikov-by-hypercentral, then all proper subgroups of* G *are Černikov-by-hypercentral*

Proof — Since finitely generated subgroups of G of finite rank are soluble-by-finite by [2] and since finitely generated Černikov-by-hypercentral are almost abelian-by-nilpotent, G is locally (soluble-by-finite). Assume, for a contradiction, that the statement is false. So G contains a proper subgroup H of finite rank which is not Černikov-by-hypercentral. We deduce by Lemma 5 that G is locally finite and by Lemma 7 that G is $\mathfrak{F}$-perfect. Suppose first that the commutator subgroup G' is properly contained in G. As noted in the proof of Lemma 5, G/G' cannot be the product of two proper subgroups and hence G/G' is quasicyclic. Since by Lemma 3 all proper normal subgroups of G are hypercentral, we deduce that $\langle G', E\rangle$ is hypercentral for all finitely generated subgroups E of G. So G is locally nilpotent and hence H is hypercentral by [13, Corollary 1 of Theorem 6.36]. This contradiction shows that G is a perfect group. Suppose that all proper normal subgroups of G have finite rank. It follows, by [3, Lemma 2.4], that G is either locally nilpotent or has a simple homomorphic image of infinite rank. Hence G must be locally nilpotent by Lemma 8, which implies, as before, the contradiction that H is hypercentral. Thus G contains a proper normal subgroup of infinite rank, say M. Since H cannot be contained in any proper subgroups of infinite rank, $G = MH$ and hence G/M is of finite rank. Since locally finite groups of finite rank are almost locally soluble by [14], we deduce that G/M is almost (hypercentral-by-soluble) by [13, Lemma 10.39], hence trivial as it is perfect and $\mathfrak{F}$-perfect, which is our final contradiction. □

The consideration of the example of Heineken and Mohamed [10] shows that a group that satisfies the hypotheses of Theorem 9 is not in general Černikov-by-hypercentral.

3 The hypercentral-by-Černikov case

In this section, we will consider the "dual" situation, that is groups in which every proper subgroup of infinite rank is hypercentral-by-Černikov. Only one preliminary result is needed. Recall that a group G is said to be periodic over a subgroup R if for every $g \in G$ there exists a positive integer n such that $g^n \in R$.

Lemma 10 *If G is a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are hypercentral-by-Černikov, then G is not simple.*

Proof — Assume, for a contradiction, that G is simple. As noted in the proof of Lemma 8, both hypercentral-by-Černikov groups and locally (soluble-by-finite) groups of finite rank are almost locally soluble, so that proper subgroups of G are almost locally soluble. We deduce that G is countable as the class of (locally soluble)-by-finite groups is countably recognizable by [4, Lemma 3.5]. By [7, Proposition 1], there exists a locally soluble subgroup R such that G is periodic over R; moreover, if the Hirsch-Plotkin radical H of R is non-trivial, then G is locally finite by [7, Theorem 2]. If $H = 1$, then R is finite since it is either hypercentral-by-soluble by [13, Lemma 10.39] or hypercentral-by-Černikov, so that G is periodic and hence locally finite. Thus, in both cases, G is locally finite. Since proper subgroups of G are (locally soluble)-by-finite, we deduce, by [11], that G is isomorphic to either $PSL(2, F)$ or $Sz(F)$, for some infinite locally finite field F. But, as noted before, each of these groups has a proper non-(hypercentral-by-Černikov) subgroup of infinite rank [12], which is our final contradiction. □

Theorem 11 *Let G be an $\mathfrak{X}$-group of infinite rank. If all proper subgroups of G of infinite rank are hypercentral-by-Černikov, then all proper subgroups of G are hypercentral-by-Černikov.*

Proof — First note that if G is finitely generated, then G has a proper subgroup of finite index and hence G is nilpotent-by-finite, which gives the contradiction that G has finite rank. Therefore finitely generated subgroups of G are either hypercentral-by-Černikov or $\mathfrak{X}$-groups of finite rank and hence soluble-by-finite. It follows that G is locally (soluble-by-finite). Assume, for a contradiction, that there is a proper subgroup of finite rank H of G which is not hypercentral-by-Černikov. So G has no proper subgroups of finite index. For, if there exists a proper normal subgroup N of finite index, then N has a normal hypercentral subgroup M such that N/M is Černikov. Since $N \leqslant N_G(M)$, M^G is the product of finitely many normal hypercentral subgroups of N and hence M^G is hypercentral. Therefore G is hypercentral-by-Černikov, a contradiction. Suppose that all proper normal subgroups of G are of finite rank. By [3, Lemma 2.4], G is either locally nilpotent or has a simple homomorphic image of infinite rank. We deduce, by Lemma 10, that G is locally nilpotent and hence H is hypercentral by [13, Corollary 1 of Theorem 6.36], a contradiction. Thus G has a proper normal subgroup L of infinite rank. Clearly, $G = HL$, so G/L is of finite rank. It follows, by [2], that G/L is almost locally soluble and hence locally soluble as G is $\mathfrak{F}$-perfect. By [13, Lemma 10.39], we deduce that G/L is hypercentral-by-soluble, which gives that G is imperfect. If G/G' is the product of two proper subgroups, then $G = AB$ is a product of two proper normal subgroups A and B which are hypercentral-by-Černikov by [8, Lemma 2]. Consequently both A and B have characteristic locally nilpotent subgroups whose factor groups are Černikov and this gives, as in the proof of Corollary 2, that $AB = G$ is (locally nilpotent)-by-Černikov. Thus by [13, Corollary 1 of Theorem 6.36], the subgroup H is hypercentral-by-Černikov, a contradiction. It follows that G/G' cannot be the product of two proper subgroups and hence G/G' is quasicyclic. Consequently, G' is hypercentral-by-Černikov. Therefore there exists a normal hypercentral subgroup C of G' such that G'/C is a Černikov group. Since C is subnormal in G, it is contained in the Hirsch-Plotkin radical of G and hence C^G is locally nilpotent. But G'/C^G and G/G' are Černikov, so G is (locally nilpotent)-by-Černikov, in particular, H is hypercentral-by-Černikov by [13, Corollary 1 of Theorem 6.36], our final contradiction. □

It is not known whenever a locally (soluble-by-finite) group of infinite rank whose proper subgroups are hypercentral-by-Černikov has to be itself hypercentral-by-Černikov.

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