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Groups with Polycyclic Non-Normal Subgroups

(Dedicated to Mario Curzio on the occasion of his 70th birthday)

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Abstract. The structure of groups in which many subgroups have a certain property χ has been investigated for several choices of the property χ. Groups whose non-normal subgroups satisfy certain finite rank conditions are studied in this article. In particular, a classification of groups in which every subgroup is either normal or polycyclic is given.

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1 Introduction

Let χ be a property pertaining to subgroups of groups. The structure of groups for which the set of all subgroups that do not have the property χ is small in some sense has been investigated for several possible choices of χ (see [3, 5] for a detailed introduction to this topic). The property χ in this context can be an absolute property (such as commutativity or nilpotency) or an embedding property (such as the property of being a normal or a subnormal subgroup). In the latter case, the first step is of course the description of groups in which χ holds for all subgroups. When χ is the property of being a normal subgroup, the situation is well known: the non-abelian groups in which all subgroups are normal are precisely those which can be decomposed as a direct product of Q_8 (the quaternion group of order 8) and any periodic abelian group without elements of order 4. Groups in which every non-abelian subgroup is normal were considered by Romalis and Sesekin [7–9], who proved that a locally soluble group with this property has finite commutator subgroup and is soluble with derived length at most 3. The hypothesis that the group is locally soluble can be weakened where it is locally graded (see [5]). Recall that a group is said to be *locally graded* if all its finitely generated non-trivial subgroups have at least one finite non-trivial homomorphic image. In particular, every locally (soluble-by-finite) group is locally graded. The consideration of Tarski groups shows that a requirement of this type is necessary when dealing with this kind of problem. More recently, Bruno and Phillips [1] have described groups in which every subgroup is either normal or locally nilpotent. The aim of this article is to study locally graded groups whose non-normal subgroups satisfy certain finite rank conditions. In particular, our main result gives a complete classification of locally graded groups with polycyclic non-normal subgroups. Clearly, all infinite subgroups of a periodic group with this property are normal, and such groups have been characterized by Černikov [2].

Most of our notation is standard and can be found in [6].

2 Statements and Proofs

Recall that a group is said to be a *Dedekind group* if all its subgroups are normal. A non-abelian Dedekind group is called *hamiltonian*.

Lemma 2.1. *Let G be a group whose non-normal subgroups are periodic. Then G is either abelian or periodic.*

Proof. Suppose G is not periodic and let a be any element of infinite order of G. Then $\langle a^8\rangle$ is a normal subgroup of G and $G/\langle a^8\rangle$ is a Dedekind group so that $(G')^2$ is contained in $\langle a^8\rangle$. It follows that a^2 does not belong to G', and hence, $[a,x]=1$ for every element x of G so that every element of infinite order of G lies in $Z(G)$. If y is any element of finite order of G, the product ay has infinite order and so y also belongs to $Z(G)$. Therefore, the group G is abelian. □

Corollary 2.2. *Let G be a locally graded group whose non-normal subgroups are locally finite. Then G is either abelian or locally finite.*

Proof. Suppose G is not abelian so that it is periodic by Lemma 2.1. Assume now by contradiction that G contains a finitely generated infinite subgroup E, and let H be any subgroup of finite index of E. Then H is not locally finite so that H is normal in G and G/H is a Dedekind group. If J is the finite residual of E, then the factor group E/J is metabelian and so finite, contradicting the hypothesis that G is locally graded. Therefore, G is a locally finite group. □

Černikov [2] showed that an infinite locally graded group whose infinite subgroups are normal is either a Dedekind group or an extension of a Prüfer group by a finite Dedekind group. Later, Phillips and Wilson [5] proved that a locally (soluble-by-finite) non-Dedekind group satisfying the minimal condition on non-normal subgroups is a Černikov group. Our next theorem is a special case of this result. On the other hand, we include its short proof for convenience.

Theorem 2.3. *Let G be a locally graded group whose non-normal subgroups are Černikov groups. Then G is either a Dedekind group or a Černikov group.*

Proof. By Corollary 2.2, it can be assumed that the group G is locally finite. Suppose G is not a Černikov group so that it does not satisfy the minimal condition on abelian subgroups (see [6, Part 1, p. 98]), and hence, contains an abelian subgroup A which is the direct product of infinitely many subgroups of prime order. Let x be any element of G, and A_1 and A_2 infinite subgroups of A such that $A_1 \cap A_2 = \langle A_1, A_2\rangle \cap \langle x\rangle = \{1\}$. Then both subgroups $\langle A_1, x\rangle$ and $\langle A_2, x\rangle$ are normal in G, and so $\langle x\rangle = \langle A_1, x\rangle \cap \langle A_2, x\rangle$ is a normal subgroup of G. Therefore, G is a Dedekind group. □

Let $\mathfrak{X}$ be a subgroup-closed class of groups and G a group whose non-normal subgroups belong to $\mathfrak{X}$. If H is any subgroup of $N = (G')^2$ which is not an $\mathfrak{X}$-group, then H is normal in G and G/H is a Dedekind group so that $H = N$. Therefore, every proper subgroup of N belongs to $\mathfrak{X}$ and so N is either an $\mathfrak{X}$-group or a minimal non-$\mathfrak{X}$-group. Our next three results provide some information on locally graded groups whose proper subgroups are soluble and satisfy certain finiteness conditions.

Lemma 2.4. *Let G be an infinite locally graded group whose proper subgroups are soluble. Then G is hyperabelian.*

Proof. First, suppose G is not finitely generated so that it is locally soluble. Then every non-abelian homomorphic image of G is not simple, and so contains a soluble non-trivial normal subgroup. It follows that every non-trivial homomorphic image of G has an abelian non-trivial normal subgroup, and hence, G is hyperabelian. Now assume G is finitely generated so that it contains a soluble normal subgroup H such that G/H is finite and H/H' is infinite. Let p be a prime which does not divide the order of G/H, and put $K/H' = (H/H')^p$. Since H is finitely generated, K is a proper subgroup of finite index of H, and the theorem of Schur–Zassenhaus applied to the finite group G/K yields that G contains a subgroup L such that $G = HL$ and $H \cap L = K$. Clearly, L is properly contained in G so that it is soluble, and also G is a soluble group. □

Let G be a group. We say that G has *finite abelian subgroup rank* if it does not contain abelian subgroups of infinite rank which are either free abelian or of prime exponent, and that G has *finite abelian section rank* if it has no infinite abelian sections of prime exponent. It is well known that any hyperabelian group with finite abelian subgroup rank also has finite abelian section rank (see [6, Part 2, p. 128]).

Corollary 2.5. *Let G be an infinite locally graded group whose proper subgroups are soluble with finite abelian section rank. Then G is a soluble group with finite abelian section rank.*

Proof. The group G is hyperabelian by Lemma 2.4. Moreover, we can assume G is not abelian so that it has finite abelian subgroup rank, and hence, also has finite abelian section rank. If G contains a proper subgroup of finite index, we obviously have that G is soluble. Finally, if G has no proper subgroups of finite index, it is well known that it is nilpotent (see [6, Part 2, Theorem 9.31]). □

Corollary 2.6. *Let G be an infinite locally graded group whose proper subgroups are polycyclic. Then G is either polycyclic or a group of type p^∞ for some prime p.*

Proof. The group G is hyperabelian by Lemma 2.4. First, suppose G has no proper subgroups of finite index, and let N be any proper normal subgroup of G. As N is polycyclic, the factor group $G/C_G(N)$ is also polycyclic (see [6, Part 1, Theorem 3.27]) so that $C_G(N) = G$ and N is contained in $Z(G)$. Therefore, $G/Z(G)$ is simple and G is abelian. Moreover, G is obviously divisible and cannot be generated by two proper subgroups so that it is a group of type p^∞ for some prime p. Now suppose G contains a proper subgroup of finite index. Then G is polycyclic-by-finite, and hence even polycyclic. □

Now we can characterize locally graded groups whose non-normal subgroups are soluble and have finite abelian section rank.

Theorem 2.7. *Let G be a locally graded group. Then every non-normal subgroup of G is a soluble group with finite abelian section rank if and only if G satisfies one of the following conditions:*

(i) *G is a Dedekind group.*

(ii) *G is a soluble group with finite abelian section rank.*

(iii) *G has finite abelian section rank and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, the commutator subgroup G' of G is finite.*

Proof. Suppose G is a non-Dedekind group whose non-normal subgroups are soluble with finite abelian section rank, and assume by contradiction that G contains an abelian subgroup A which is a direct product of infinitely many isomorphic cyclic groups, which are either infinite or have prime order. Then $A = A_1 \times A_2$, where both A_1 and A_2 have infinite rank so that A_1 and A_2 are normal in G and the factor groups G/A_1 and G/A_2 are Dedekind groups. Since G is not abelian, at least one of the groups G/A_1 and G/A_2

is hamiltonian so that G is periodic and A has prime exponent. Moreover, G is nilpotent and its commutator subgroup has exponent 2 so that $G = G_2 \times G_{2'}$, where G_2 is a 2-group and $G_{2'}$ is an abelian $2'$-group. Then G_2 is a non-Dedekind group whose non-normal subgroups are Černikov groups, and hence, G_2 is a Černikov group by Theorem 2.3. Since $G/G_{2'}$ is not a Dedekind group, the subgroup $G_{2'}$ has finite abelian section rank, and hence, G itself has finite abelian section rank. This contradiction shows that G has finite abelian subgroup rank. If G is soluble, it is well known that G also has finite abelian section rank (see [6, Part 2, p. 128]). Now suppose G is not soluble so that $N = (G')^2$ is a non-soluble group whose proper subgroups are soluble with finite abelian section rank. Then it follows from Corollary 2.5 that N is finite. Moreover, G/N is a Dedekind group. Clearly, N is not contained in its centralizer $C_G(N)$ so that $G/C_G(N)$ is not a Dedekind group, and $C_G(N)$ must have finite abelian section rank. Since $G/C_G(N)$ is finite, the group G itself has finite abelian section rank.

Conversely, suppose the group G satisfies condition (iii) and let H be a non-normal subgroup of G. Then N is not contained in H so that $H \cap N$ is soluble, and H is a soluble group with finite abelian section rank. □

The direct product of the alternating group A_5 with any hamiltonian group with finite abelian section rank is a group satisfying condition (iii) of Theorem 2.7 in which the commutator subgroup is not minimal non-soluble.

Corollary 2.8. *Let G be a locally graded group. Then every non-normal subgroup of G is a soluble group with finite Prüfer rank if and only if G satisfies one of the following conditions*:

(i) *G is a Dedekind group.*

(ii) *G is a soluble group with finite Prüfer rank.*

(iii) *G has finite Prüfer rank and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, the commutator subgroup G' of G is finite.*

Proof. Clearly, we only need to show that, if G is a non-Dedekind group whose non-normal subgroups are soluble with finite Prüfer rank, then G is either a soluble group with finite Prüfer rank or satisfies condition (iii) of the statement. It follows from Theorem 2.7 that G has finite abelian section rank, and G is either soluble or contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. Assume by contradiction that G has infinite Prüfer rank. Then G contains an abelian subgroup A with infinite Prüfer rank (see [6, Part 2, p. 89]) so that the subgroup consisting of all elements of finite order of A also has infinite Prüfer rank, and hence, A can be chosen to be periodic. Thus, $A = A_1 \times A_2$, where both A_1 and A_2 have infinite Prüfer rank. Then A_1 and A_2 are normal in G, and the factor groups G/A_1 and G/A_2 are Dedekind groups which cannot be both abelian. Suppose G/A_1 is hamiltonian. Clearly, there exists in A a properly descending chain $A_1 = B_1 > B_2 > \cdots > B_n > \cdots$ of subgroups with infinite Prüfer rank such that $\cap_{n\geq 1} B_n = \{1\}$. Each subgroup B_n is normal in G and the group G/B_n is hamiltonian so that $G'/(G' \cap B_n)$ has order 2. It follows that

$$G' \cap B_1 = G' \cap B_2 = \cdots = G' \cap B_n = \cdots ,$$

and hence, G' has order 2. Therefore, G is nilpotent and $G = G_2 \times G_{2'}$, where G_2 is a 2-group and $G_{2'}$ is an abelian $2'$-group. Then G_2 is a Černikov group which is not a Dedekind group so that $G_{2'}$ has finite Prüfer rank, and G itself has finite Prüfer rank. This contradiction shows that G has finite Prüfer rank and completes the proof of the corollary. □

A soluble group G is called an $\mathfrak{S}_1$*-group* if it has finite abelian section rank and $\pi(G)$ is finite, where $\pi(G)$ denotes the set of all prime numbers that are orders of elements of G. Note that a periodic group G is an $\mathfrak{S}_1$-group if and only if it is a soluble Černikov group (see [6, Part 2, Theorem 10.33]).

Corollary 2.9. *Let G be a locally graded group. Then every non-normal subgroup of G is an $\mathfrak{S}_1$-group if and only if G satisfies one of the following conditions*:

(i) *G is a Dedekind group.*

(ii) *G is an $\mathfrak{S}_1$-group.*

(iii) *G is a finite extension of an $\mathfrak{S}_1$-group and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, the commutator subgroup G' of G is finite.*

Proof. Suppose G is a non-Dedekind group whose non-normal subgroups are $\mathfrak{S}_1$-groups so that G is a soluble-by-finite group with finite abelian section rank by Theorem 2.7. Moreover, if G is not soluble, then it contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. If T is the largest periodic normal subgroup of G, then the set of primes $\pi(G/T)$ is finite (see [6, Part 2, Lemma 9.34]). Assume the locally finite group T does not satisfy the minimal condition on subgroups. Then T contains an abelian subgroup A with infinitely many non-trivial primary components. Write $A = A_1 \times A_2$, where A_1 and A_2 are subgroups such that the sets $\pi(A_1)$ and $\pi(A_2)$ are infinite. Then A_1 and A_2 are normal in G, and G/A_1 and G/A_2 are Dedekind groups. Since G is not abelian, it follows that at least one of such groups is hamiltonian so that G is periodic, and hence, it is a Černikov group by Theorem 2.3. This contradiction shows that the set $\pi(G)$ is finite, and so G satisfies either (ii) or (iii). The converse statement is obvious. □

A group G is said to be *minimax* if it has a series of finite length whose factors satisfy either the minimal or maximal condition on subgroups. In particular, if G is an extension of a group satisfying the minimal condition by a group satisfying the maximal condition on subgroups, we say that G is a *Min-by-Max group*. Clearly, every soluble minimax group is an $\mathfrak{S}_1$-group.

Corollary 2.10. *Let G be a locally graded group. Then every non-normal subgroup of G is a soluble minimax group if and only if G satisfies one of the following conditions*:

(i) *G is a Dedekind group.*

(ii) *G is a soluble minimax group.*

(iii) *G is minimax and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, the commutator subgroup G' of G is finite.*

Proof. We only need to show that, if G is a non-Dedekind group whose non-normal subgroups are soluble and minimax, then either G itself is a soluble minimax group or it satisfies condition (iii) of the statement. It follows from Corollary 2.9 that G is an $\mathfrak{S}_1$-group, and it is either soluble or contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, we may suppose G is not periodic. Assume by contradiction that G is not minimax so that it contains an abelian subgroup A which is not minimax (see [6, Part 2, Theorem 10.35]). Let T be the subgroup consisting of all elements of finite order of A. Then the set of primes $\pi(T)$ is finite by Corollary 2.9 so that T is a Černikov group and $A = T \times B$, where B is a torsion-free subgroup which is not minimax. Let E be a finitely generated subgroup of B such that B/E is periodic. Clearly, B/E is not a Černikov group so that, for each positive integer $n \geq 3$, we have $B/E^{2^n} = B_{1,n}/E^{2^n} \times B_{2,n}/E^{2^n}$, where both $B_{1,n}/E^{2^n}$ and $B_{2,n}/E^{2^n}$ have infinitely many primary components. Therefore, $B_{1,n}$ and $B_{2,n}$ are not minimax so that they are normal in G and the factor groups $G/B_{1,n}$ and $G/B_{2,n}$ are Dedekind groups. Thus, $(G')^2 \leq B_{1,n} \cap B_{2,n} = E^{2^n}$ so that $(G')^2 \leq \cap_{n\geq 3} E^{2^n} = \{1\}$ and G' is periodic. On the other hand, at least one of the groups $G/B_{1,n}$ and $G/B_{2,n}$ contains an element of order 8, and hence is abelian so that G' is torsion-free. It follows that $G' = \{1\}$ and G is abelian. This last contradiction completes the proof of the corollary. □

Lemma 2.11. *Let G be a locally graded group whose non-normal subgroups are polycyclic. Then G is either polycyclic-by-finite or metabelian.*

Proof. Suppose G is neither polycyclic-by-finite nor a Dedekind group so that it is a soluble-by-finite group with finite Prüfer rank by Corollary 2.8. Then G contains an abelian subgroup A which is not finitely generated (see [6, Part 1, Theorem 3.31]). Moreover, A is normal in G and G/A is a Dedekind group so that G is soluble. Clearly, we can assume the group G/A is hamiltonian. If A is periodic, it follows that G is a periodic group whose infinite subgroups are normal and then G is either a Dedekind group or a finite extension of a Prüfer group by a finite Dedekind group (see [2, Theorem 1.3]). In this case, it follows immediately that G is metabelian.

This argument allows us to assume all periodic abelian subgroups of G are finite so that the Hall–Kulatilaka–Kargapolov theorem yields that every periodic subgroup of G is finite. In particular, the subgroup A can be chosen to be torsion-free. Since A has finite Prüfer rank, $G/C_G(A)$ is isomorphic to a periodic linear group over the field of rational numbers, and hence is finite. Let B be a finitely generated subgroup of A such that A/B is periodic. Then the normal closure $H = B^G$ is also finitely generated, and for each prime number p, G/H^p is a periodic group whose infinite subgroups are normal so that G/H^p is metabelian. As $\cap_p H^p = \{1\}$, it follows that G is also metabelian. □

Lemma 2.12. *Let A be a torsion-free abelian group containing a finitely generated subgroup B such that A/B is a group of type 2^∞. If A is 2-divisible, then A is isomorphic to the additive group $\mathbb{Q}_2$ consisting of all rational numbers whose denominators are powers of 2.*

Proof. Assume B is not cyclic so that it contains a subgroup B_1 such that B/B_1 is a Klein 4-group, and the periodic group A/B_1 has an image of order 2. This contradiction shows that B is cyclic so that A has rank 1, and hence, it is isomorphic to a subgroup of the additive group of rational numbers. It follows immediately that A is isomorphic to $\mathbb{Q}_2$. □

Now it is possible to prove the main result of this paper.

Theorem 2.13. *Let G be a locally graded group. Then every non-normal subgroup of G is polycyclic if and only if G satisfies one of the following conditions:*

(i) *G is a Dedekind group.*

(ii) *G is polycyclic.*

(iii) *G is an extension of a Prüfer group by a finite Dedekind group.*

(iv) *G is a soluble minimax group whose finite residual J is a group of type p^∞ for some prime p and contains G'. Moreover, every abelian subgroup of G is Min-by-Max.*

(v) *$G = M \times E$, where M is isomorphic to the additive group $\mathbb{Q}_2$ of all rational numbers whose denominators are powers of 2 and E is a finite hamiltonian group.*

(vi) *G is polycyclic-by-finite and contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group. In particular, the commutator subgroup G' of G is finite.*

Proof. Let G be a locally graded non-Dedekind group whose non-normal subgroups are polycyclic. If G is periodic, then all its infinite subgroups are normal and so G satisfies condition (iii) (see [2, Theorem 1.3]). On the other hand, if G is polycyclic-by-finite, then Theorem 2.7 yields that G is either polycyclic or contains a finite normal minimal non-soluble subgroup N such that G/N is a Dedekind group so that G satisfies either (ii) or (vi) of the statement. Suppose G is neither periodic nor polycyclic-by-finite so that it is metabelian by Lemma 2.11. Clearly, we can assume G is not a Dedekind group so that G is minimax by Corollary 2.10, and in particular, all its periodic subgroups are Černikov groups. First, suppose the commutator subgroup G' of G is not finitely generated and let H be any subgroup of finite index of G'. Then H is not finitely generated so that it is normal in G and G/H is a Dedekind group. It follows that G'/H has order at most 2. Thus, the finite residual R of G' has index at most 2 in G' so that R is divisible. Moreover, every proper subgroup of R is finitely generated so that R is a group of type p^∞ for some prime p. If $R \neq G'$, the group G/R is hamiltonian so that G is periodic, a contradiction. Therefore, $G' = R$ is a group of type p^∞. Let J be the finite residual of G. Then J is a periodic divisible abelian group (see [6, Part 2, Theorem 10.33]) and $J = G' \times J_1$ for some divisible subgroup J_1. If $J_1 \neq \{1\}$, it is normal in G and G/J_1 is a Dedekind group, which is impossible as G' is infinite. It follows that $J_1 = \{1\}$ and $G' = J$ is the finite residual of G. Let A be any abelian subgroup of G. Then $A = A_1 \times A_2$, where A_1 is a Černikov group and A_2 is torsion-free. If A_2 is not finitely generated, it is normal in G and G/A_2 is a Dedekind group. This contradiction shows that A_2 is finitely generated so that A is Min-by-Max and G satisfies condition (iv) of the statement. Now suppose G' is finitely generated so that G/G' is not finitely generated and there exists in G' a collection $\{K_i\}_{i\in I}$ of G-invariant subgroups of finite index such that $\cap_{i\in I} K_i = \{1\}$. For each $i \in I$, the minimax group $\bar{G}_i = G/K_i$ has finite commutator subgroup so that $\bar{G}_i/Z(\bar{G}_i)$ is also finite. Then $Z(\bar{G}_i)$ is not finitely generated, and hence, $\bar{G}_i/Z(\bar{G}_i)$ is a Dedekind group. It follows that $\bar{G}_i$ is nilpotent with class at most 3 for every $i \in I$ so that G is also a nilpotent group. Let T be the subgroup consisting of all elements of finite order of G. If G' is infinite, the factor group G/T has infinite commutator subgroup so that it is not a Dedekind group, and hence, T must be finite. Then G is residually finite. Moreover, every finite homomorphic image of G is a Dedekind group and so G' has exponent 2. This contradiction proves that G' is finite so that $G/Z(G)$ is also finite. Since G is a minimax group, we have $Z(G) = U \times V$, where U is periodic and V is torsion-free. Suppose V is not finitely generated so that G/V is hamiltonian and G' has order 2. If $V^2 \neq V$, then $V^{2^{n+1}} \neq V^{2^n}$ for each n so that G/V^8 cannot be hamiltonian, and hence, it is abelian, a contradiction. Thus, $V^2 = V$ and V contains a subgroup W such that V/W is isomorphic to $Z(2^\infty)$. Clearly, G/W is not a Dedekind group, and hence, W must be finitely generated so that V is isomorphic to $\mathbb{Q}_2$ by Lemma 2.12. Assume U is infinite and let U_0 be the finite residual of U. Since the group G/V is hamiltonian, U cannot contain subgroups of type 2^∞ so that $U_0 \cap G' = \{1\}$ and the non-periodic group G/U_0 should be hamiltonian, a contradiction. Therefore, U is finite so that T is also finite and $G/G' = L/G' \times T/G'$, where L/G' is torsion-free. Since G is a finite extension of a 2-divisible subgroup, the group L/G' is also 2-divisible so that L is abelian and $L = M \times G'$ for some torsion-free 2-divisible subgroup M. Therefore, $G = LT = MG'T = MT$ and $M \cap T = \{1\}$. Moreover, M is normal in G since it is not polycyclic, and hence, $G = M \times T$, where T is a finite hamiltonian group. Furthermore, the same argument used above proves that the abelian group M is isomorphic to $\mathbb{Q}_2$ so that G satisfies condition (v). Finally, suppose V is finitely generated so that T is an infinite Černikov group and G is Min-by-Max. Let P be any Prüfer subgroup of G. Then P is normal in G and the non-periodic group G/P is abelian so that G' is contained in P. It follows that the finite residual of G is a group of type p^∞ for some prime p and G satisfies (iv).

Conversely, it is clearly enough to show that, if G satisfies one of the conditions (iv) and (v), then every non-normal subgroup of G is polycyclic. Let X be any subgroup of G which is not polycyclic. First, suppose G satisfies (iv). If G' is not contained in X, the intersection $X \cap G'$ is finite so that X' is finite, and so $X/Z(X)$ is also finite. Then $Z(X)$ is a Min-by-Max group which is not finitely generated so that the finite residual J of G lies in $Z(X)$, a contradiction since G' is contained in J. Therefore, $G' \leq X$ and X is normal in G. Now suppose G satisfies (v). Then $X \cap M$ is not finitely generated, and hence, $M/(X \cap M)$ is a finite abelian group of odd order. It follows that the group $G/(X \cap M)$ is hamiltonian so that X is normal in G. □

Finally, observe that an example of a group, which is not Min-by-Max and satisfies condition (iv) in Theorem 2.13, is given in Example 1 of [4].

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