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Groups with many subgroups having a transitive normality relation

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Abstract. A group is said to be a T-group if all its subnormal subgroups are normal. The structure of groups satisfying the minimal condition on subgroups that do not have the property T is investigated. Moreover, locally soluble groups with finitely many conjugacy classes of subgroups which are not T-groups are characterized.

Keywords: T-group; Černikov group.

1 Introduction

A group G is said to be a *T-group* if every subnormal subgroup of G is normal, i.e. if normality in G is a transitive relation. The structure of finite soluble T-groups has been described by Gaschütz [5], while Robinson [6] investigated infinite soluble groups with the property T. It turns out in particular that every subgroup of a finite soluble T-group is likewise a T-group, a result that is no longer true for infinite soluble T-groups. We shall say that a group G is a *$\bar{T}$-group* if all its subgroups are T-groups. Finite groups which do not have the property T but whose proper subgroups are T-groups (i.e. finite minimal-non-T groups) have been classified by Robinson in [7], where he also proved that locally finite minimal-non-T groups are finite.

Recently many authors have investigated the behaviour of groups for which the set of all subgroups which do not have a given property is small in some sense. In particular, S.N. Černikov [3] described groups satisfying the minimal condition on non-abelian subgroups, while groups with the minimal condition on subgroups that do not have the property FC have been studied in [4] (recall here that a group G is an *FC-group* if every element of G has finitely many

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conjugates). The aim of this article is to consider the corresponding problem for the property T, and our main result is the following.

Theorem A. *Let G be a group having no infinite simple sections. If G satisfies the minimal condition on subgroups which are not T-groups, then either G is a Černikov group or it is a soluble $\bar{T}$-group.*

The above theorem can be used to study groups in which subgroups that do not have the property T fall into finitely many conjugacy classes.

Theorem B. *Let G be a locally soluble group having finitely many conjugacy classes of subgroups which are not T-groups. Then either G is finite or it is a $\bar{T}$-group.*

Most of our notation is standard and can for instance be found in [8].

2 Proofs

In our proofs we will use many results concerning the structure of soluble T-groups, for which we refer to [6]. If G is any T-group, we have $G^{(2)} = G^{(3)}$ (that is $G^{(2)}$ is the last term of the derived series of G) and hence soluble T-groups are metabelian. Moreover, every soluble T-group is locally supersoluble, so that in particular finite soluble T-groups are supersoluble and finite minimal-non-T groups are soluble. Recall also that finitely generated soluble T-groups either are finite or abelian, and that a group G has the property T if and only if every finite subset of G is contained in a T-subgroup of G.

Our first result proves that minimal-non-T groups are finite, provided that they satisfy a suitable condition, imposed in order to avoid Tarski groups and other similar pathologies.

Proposition 1. *Let G be an infinite group whose finitely generated subgroups have no infinite simple sections. If every proper subgroup of G is a T-group, then also G is a T-group.*

Proof. Assume by contradiction that G is not a T-group. Then there exists a finitely generated subgroup of G which is not a T-group, and hence G itself must be finitely generated. Therefore the group G has no infinite simple sections, and hence there exists a properly descending chain

$$G_1 > G_2 > \ldots > G_n > \ldots$$

consisting of normal subgroups of finite index of G. Every G/G_n is a finite group whose proper subgroups are T-groups, so that it is soluble, and has derived length at most 3 since soluble T-groups are metabelian. It follows that $G^{(3)}$ is contained in every G_n, and $\bar{G} = G/G^{(3)}$ is an infinite finitely generated soluble group whose proper subgroups are T-groups. In particular $\bar{G}$ is polycyclic. Let $\bar{H}$ be any proper subgroup of $\bar{G}$, and let $\bar{K}$ be a proper subgroup of finite index of $\bar{G}$ containing $\bar{H}$. Then $\bar{K}$ is an infinite finitely generated soluble T-group, and hence it is abelian, so that also $\bar{H}$ is abelian. Therefore every proper subgroup of $\bar{G}$ is abelian, and then it is well-known that also $\bar{G}$ is abelian. Thus $G' = G^{(3)}$ is a perfect non-trivial group. Moreover, G' is the normal closure of a finite subset of G, so that it contains a normal subgroup N of G such that G'/N is a chief factor of G. Assume that G'/N is not simple, so that there exists a normal subgroup X of G' such that $N < X < G'$. Then X is subnormal in G, and hence it is normal in every proper subgroup of G containing G'. It follows that G/G' has a unique maximal subgroup, and hence it is a finite cyclic group. This contradiction proves that G'/N is simple, so that it is finite and G/N is soluble-by-finite. On the other hand, we have already shown that every finite homomorphic image of G is soluble, so that G/N is soluble, and this last contradiction proves that G is a T-group. □

A group class $\mathfrak{X}$ is said to be *S_n-closed* if every normal (and hence also every subnormal) subgroup of an $\mathfrak{X}$-group is likewise an $\mathfrak{X}$-group. Recall also that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Clearly, every group having no infinite simple sections is locally graded.

Lemma 2. *Let $\mathfrak{X}$ be an S_n-closed class of groups, and let G be a locally graded group in which every proper subgroup either is a Černikov group or belongs to $\mathfrak{X}$. Then either G is a Černikov group or all its proper normal subgroups are $\mathfrak{X}$-groups.*

Proof. Suppose that G contains a proper normal subgroup N which is not an $\mathfrak{X}$-group. Then every proper subgroup of G containing N is a Černikov group. In particular, N is a Černikov group and all proper subgroups of G/N are Černikov groups. It follows that the locally graded group G satisfies the minimal condition on subgroups, so that it is locally finite. Therefore G is a Černikov group (see for instance [8] Part 1, p.98). □

The proof of Theorem A will be accomplished through a series of lemmas dealing with the structure of groups in which every subgroup either is a Černikov group or a T-group.

Lemma 3. *Let G be a group in which every proper subgroup either is a Černikov group or a T-group. If G has no infinite simple sections, then every minimal normal subgroup of G is finite.*

Proof. Assume by contradiction that G contains an infinite minimal normal subgroup N. Then N is not a Černikov group, so that it is a T-group. Clearly N is not simple, and hence it has a proper non-trivial normal subgroup X. Let g be an element of G such that $X^g \neq X$. Then the subgroup $\langle g, N \rangle$ is not a T-group, so that $\langle g, N \rangle = G$. Moreover, X is normal in every proper subgroup of G containing N, so that G/N has a unique maximal subgroup, and hence it is finite. Therefore G satisfies the minimal condition on normal subgroups, and so also N satisfies the minimal condition on normal subgroups (see [8] Part 1, Theorem 5.21). Let M be a minimal normal subgroup of N. Then M is simple and hence finite. Moreover, M has finitely many conjugates in G, so that also $N = M^G$ is finite by Dietzmann's Lemma (see [8] Part 1, p.45). This contradiction proves the lemma. □

Lemma 4. *Let G be a group in which every proper subgroup either is a Černikov group or a T-group. If G has no infinite simple sections, then it is locally (soluble-by-finite).*

Proof. Clearly it can be assumed that G is a finitely generated infinite group. Then G contains a maximal normal subgroup N, and the factor group G/N is finite. Clearly N is not a Černikov group, so that every proper subgroup of G/N is a T-group, and G/N is soluble with derived length at most 3. Therefore $G/G^{(3)}$ is a finitely generated soluble non-trivial group. Assume that $G^{(3)}$ is not a Černikov group, so that every proper subgroup of $G/G^{(3)}$ is a T-group, and Proposition 1 yields that $G/G^{(3)}$ either is finite or abelian. Thus $G/G^{(3)}$ is finitely presented, and hence $G^{(3)}$ is the normal closure of a finite subset of G. Let M be a maximal proper G-invariant subgroup of $G^{(3)}$. Then $G^{(3)}/M$ is finite by Lemma 3, so that G/M is soluble-by-finite, and so even soluble. Since M is not a Černikov group, G/M has derived length at most 3. This contradiction shows that $G^{(3)}$ is a Černikov group, and hence G is soluble-by-finite. □

Lemma 5. *Let G be a perfect group in which every proper subgroup either is a Černikov group or a T-group. If G has no infinite simple sections, then it satisfies the minimal condition on normal subgroups.*

Proof. Assume by contradiction that

$$G = H_0 > H_1 > \ldots > H_n > \ldots$$

is a properly descending chain of normal subgroups of G. Since H_1 is not a Černikov group, every proper subgroup of G/H_1 is a T-group. It follows from Proposition 1 that G/H_1 either is finite or a $\bar{T}$-group. Moreover, G is locally (soluble-by-finite) by Lemma 4, so that G/H_1 is soluble and G' is properly contained in G. This contradiction proves that G satisfies the minimal condition on normal subgroups. □

Lemma 6. *Let G be a group in which every proper subgroup either is a Černikov group or a T-group. If G is hyper-(abelian or finite), then either G is a Černikov group or it is soluble.*

Proof. Assume that G is not a Černikov group, and let H/K be any finite normal section of G. Then $C = C_G(H/K)$ is a normal subgroup of G and G/C is finite. Clearly C is not a Černikov group, so that every proper subgroup of G/C is a T-group, and hence G/C is soluble with derived length at most 3. Thus $G^{(3)}$ acts trivially on every finite normal section of G. Let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < \ldots < G_\tau = G$$

be an ascending normal series of G whose factors either are abelian or finite. Then

$$\begin{aligned}\{1\} = G_0 \cap G^{(3)} \leq G_1 \cap G^{(3)} \leq \ldots \leq G_\tau \cap G^{(3)} = G^{(3)} \\ \leq G^{(2)} \leq G^{(1)} \leq G\end{aligned}$$

is an ascending normal series with abelian factors, and so G is hyperabelian. Assume that every proper normal subgroup of G is a Černikov group. Then G satisfies the minimal condition on normal subgroups, and hence it is hyperfinite by Lemma 3, so that G is a Černikov group (see [8] Part 1, p.148). This contradiction shows that G contains a proper normal subgroup N which is not a Černikov group. Then every proper subgroup of G/N is a T-group, so that G/N itself is a T-group by Proposition 1, and hence it is soluble. It follows that G' is a proper subgroup of G, so that G' is a T-group by Lemma 2 and G is soluble. □

Lemma 7. *Let G be a periodic hyperabelian group in which every proper subgroup either is a Černikov group or a T-group. Then either G is a Černikov group or it is a $\bar{T}$-group. In particular, G is soluble.*

Proof. Suppose that G is not a Černikov group, so that every proper normal subgroup of G is a T-group by Lemma 2. Let F be the Fitting subgroup of G, and let E be any finite subgroup of F. Then E is subnormal in G and E^G is a proper normal subgroup of G, so that E^G is a T-group, and E has defect at most 2. In particular, F is nilpotent, and hence all its proper subgroups are T-groups. Therefore F itself is a T-group, and so even a Dedekind group. Since G is a periodic hyperabelian group, F is not a Černikov group, and so it contains an infinite G-invariant subgroup A which is a direct product of subgroups of prime order. Let X be any finite subgroup of G, and let A_1 be a subgroup of finite index of A such that $A_1 \cap X = \{1\}$. Then also the subgroup

$$A_2 = \bigcap_{x \in X} A_1^x$$

has finite index in A, and so it contains a proper subgroup A_3 of finite index. Put

$$A_4 = \bigcap_{x \in X} A_3^x,$$

so that the index $|A : A_4|$ is finite and A_4 is an infinite X-invariant subgroup. Clearly $XA_4 < XA_2$, and hence XA_4 is a proper subgroup of G which is not a Černikov group. Therefore XA_4 is a T-group, so that also $X \simeq XA_4/A_4$ is a T-group. We have shown that every finite subgroup of G is a T-group, and it follows from this property that G is a $\bar{T}$-group. □

Lemma 8. *Let G be a periodic group in which every proper subgroup either is a Černikov group or a T-group. If G has no infinite simple sections, then either G is a Černikov group or it is a soluble $\bar{T}$-group.*

Proof. Suppose that G is not a Černikov group, and assume that $G^{(n)} = G^{(n+1)}$ for some non-negative integer n. Then $G^{(n)}$ satisfies the minimal condition on normal subgroups by Lemma 5, and hence it is hyperfinite by Lemma 3. Application of Lemma 6 yields now that $G^{(n)}$ is soluble-by-finite, so that also G is soluble-by-finite. Let N be a soluble normal subgroup of finite index of G. Then N is not a Černikov group, so that all proper subgroups of the finite group G/N are T-groups, and hence G/N is soluble. It follows that G is soluble, and so it is a $\bar{T}$-group by Lemma 7. □

Proof of Theorem A. Assume by contradiction that G contains subgroups which neither are Černikov groups nor T-groups, and let H be a minimal element of the set of such subgroups. Then every proper subgroup of H either is a Černikov group or a T-group, and it follows from Lemma 8 that H is not periodic. Let a be an element of infinite order of H, and let E be a finitely generated subgroup of H which is not a T-group. The finitely generated infinite group $\langle a, E\rangle$ is soluble-by-finite by Lemma 4, so that it is even soluble, and hence it is not a T-group. Therefore $H = \langle a, E\rangle$ is a finitely generated soluble group. Let X be any proper subgroup of finite index of H. Then X is a finitely generated infinite soluble T-group, and so is abelian. In particular, H is polycyclic, so that every proper subgroup of H is contained in a proper subgroup of finite index, and hence all proper subgroups of H are abelian. Therefore H itself is abelian, and this contradiction shows that every subgroup of G either is a Černikov or a T-group. Then G is locally (soluble-by-finite) by Lemma 4. If G is periodic, it follows from Lemma 8 that either G is a Černikov group or a soluble $\bar{T}$-group. Suppose now that G contains an element g of infinite order, and let K be any finitely generated subgroup of G. Then $\langle g, K\rangle$ is a finitely generated infinite soluble-by-finite group, so that $\langle g, K\rangle$ is a soluble T-group, and hence it is abelian. Therefore also G is abelian, and the theorem is proved. □

Let $\mathfrak{X}$ be a group theoretical property. There is often a strong connection between groups with finitely many conjugacy classes of $\mathfrak{X}$-subgroups and groups satisfying the minimal condition on $\mathfrak{X}$-subgroups. This is actually a consequence of the following result of D.I. Zaicev, for a proof of which we refer to [1], Lemma 4.6.3.

Lemma 9. *Let G be a group locally satisfying the maximal condition on subgroups. If H is a subgroup of G and $H^x \leq H$ for some element x of G, then $H^x = H$.*

Proof of Theorem B. Assume that the group G is not soluble, so that for each positive integer n it contains a finitely generated subgroup E_n with derived length at least n. Since every soluble T-group is metabelian, the set $\{E_n \mid n \in \mathbb{N}\}$ contains infinitely many pairwise non-isomorphic subgroups of G that are not T-groups. This contradiction shows that G is soluble. On the other hand, every soluble T-group is locally supersoluble, and hence G has finitely many conjugacy classes of subgroups which do not locally satisfy the maximal condition. Therefore G locally satisfies the maximal condition on subgroups (see [4], Proposition 3.3), and it follows from Lemma 9 that G satisfies both the minimal and the maximal condition on subgroups which are not T-groups. Application of Theorem A yields that either G is a Černikov group or it is a $\bar{T}$-group. Suppose that G is not a $\bar{T}$-group, so that it is a Černikov group and contains a subgroup H which is minimal-non-T. Assume now that G is infinite. Clearly H is finite, and hence there exists a properly ascending chain

$$H = H_1 < H_2 < \ldots < H_n < \ldots$$

consisting of finite subgroups of G, so that H_n is a T-group for some positive integer n. Since every finite soluble T-group is a $\bar{T}$-group, we obtain that also H is a T-group, and this contradiction completes the proof of the theorem. □

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