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GROUPS WHOSE SUBGROUPS HAVE SMALL AUTOMIZERS

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In Memoriam Valeria Fedri

If H is a subgroup of a group G, the *automizer* of H in G is the group of all automorphisms of H induced by elements of its normalizer $N_G(H)$. The subgroup H is said to have *small* automizer if $Aut_G(H) = Inn(H)$, i.e. if $N_G(H) = HC_G(H)$. This article is devoted to the study of groups for which many subgroups have small automizer.

1. Introduction.

Let G be a group and let H be a subgroup of G. The elements of the normalizer $N_G(H)$ of H induce a group of automorphisms of H, which is called the *automizer* $Aut_G(H)$ of H in G. We clearly have

$$Inn(H) \leq Aut_G(H) \leq Aut(H).$$

The automizer $Aut_G(H)$ is called *large* if $Aut_G(H) = Aut(H)$ and *small* if $Aut_G(H) = Inn(H)$. The latter condition is clearly equivalent to $N_G(H) = HC_G(H)$.

Groups in which all subgroups have large automizers were called *MD-groups*, and have been studied in [3], [6] and [10]. In [1], finite

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groups with large automizers of their abelian subgroups were characterized, while in [4] it was shown that finite groups with large automizers of their non-abelian subgroups are soluble.

Here, we are interested in the "small" case. This was first considered by H. Zassenhaus [11] in his proof of a theorem of MacLagan-Wedderburn (see [2] for a short proof and more information). Our main result on the embedding of abelian subgroups is:

THEOREM A.

(a) *Let G be an infinite group having an ascending series with abelian factors. If $N_G(A) = C_G(A)$ for every infinite abelian ascendant subgroup A of G, then G is abelian.*

(b) *Let G be a group having an ascending series with locally (soluble-by-finite) factors. If $N_G(A) = C_G(A)$ for every abelian subgroup A of G, then G is abelian.*

Theorem A is an extension of the aforementioned result of Zassenhaus to infinite groups with certain solubility conditions. In fact, as the Tarski groups have small automizers of their abelian subgroups, some extra hypothesis on G is necessary. It will also be proved that, if G is a hyperabelian group whose normal subgroups have small automizers, then G is abelian.

The objective of Section 3 is the investigation of infinite *SANS-groups*, which are defined as the groups in which the automizers of all non-abelian subgroups are small (see [2]). Again, the existence of Tarski groups shows that we have to introduce some extra assumptions on infinite *SANS*-groups in order to get sensible results. We shall prove that, if G is a locally soluble non-abelian group with small automizers of its non-abelian subgroups, then G contains an abelian normal subgroup of prime index. Moreover, if G is a finitely generated soluble *SANS*-group, then it is finite-by-abelian, and so even central-by-finite.

For certain classes of groups, we have a complete characterization of the property *SANS*. In particular, it will be proved that a locally nilpotent non-abelian group G is *SANS* if and only if the factor group $G/Z(G)$ has order p^2, where p is a prime. Moreover, for locally finite groups the following theorem holds, which is an extension of results in [2].

THEOREM B. *Let G be a locally finite non-nilpotent group. Then G is an $SANS$-group if and only if one of the following conditions holds:*

(a) *The Fitting subgroup* $FitG$ *of* G *has prime index, and all Sylow subgroups of* G *are abelian.*

(b) $G = A \times K$ *where* A *is abelian and* $K \cong PSL(2, p^f)$ *is a simple group such that* $(p^f - 1)/h$ *is a prime where* $h = (p-1, 2)$.

Our last result contains a characterization of all locally soluble *SANS*-groups. In particular, it turns out that every torsion-free locally soluble *SANS*-group is abelian.

THEOREM C. *Let* G *be a locally soluble non-nilpotent group. Then* G *has the property SANS if and only if it contains an abelian normal subgroup* A *of prime index and there exists a torsion-free subgroup* B *of* $Z(G)$ *such that* G/B *is a periodic group with all Sylow subgroups abelian.*

All notation used in this paper is standard and can be found in [5] or [9].

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2. Small automizers of abelian subgroups.

Our first lemma deals with the behaviour of hypercentral subgroups of groups whose infinite abelian subgroups have small automizers.

LEMMA 2.1. *Let* G *be a group such that* $N_G(A) = C_G(A)$ *for every infinite abelian ascendant subgroup* A. *Then every infinite hypercentral ascendant subgroup of* G *is contained in the centre of* G.

Proof. Let H be an infinite hypercentral ascendant subgroup of G, and consider a maximal abelian normal subgroup A of H. Then $C_H(A) = A$, so that A is infinite and hence

$$H = N_H(A) = C_H(A) = A.$$

Therefore the subgroup H is abelian. Let

$$H = H_0 \leq H_1 \leq \ldots \leq H_\alpha \leq H_{\alpha+1} \leq \ldots \leq H_\tau = G$$

be an ascending series, and let $\alpha \leq \tau$ be an ordinal such that H is contained in $Z(H_\beta)$ for every $\beta < \alpha$. If α is a limit ordinal, it follows immediately that H is also contained in $Z(H_\alpha)$. Suppose that α is not a limit ordinal, so that H lies in $Z(H_{\alpha-1})$. Since the abelian subgroup $Z(H_{\alpha-1})$ is ascendant in G, we have $N_G(Z(H_{\alpha-1})) = C_G(Z(H_{\alpha-1}))$, and hence $H \leq Z(H_{\alpha-1}) \leq Z(H_\alpha)$. Therefore the subgroup H is also contained in $Z(G)$. □

Recall that the *Gruenberg radical* of a group G is the subgroup generated by all abelian ascendant subgroups of G. It is easy to show that if G is an infinite group having an ascending series with abelian factors, then also the Gruenberg radical of G is infinite.

Proof of Theorem A - (a) Assume that the group G is not abelian, so that the factor group $G/Z(G)$ contains an abelian non-trivial ascendant subgroup $A/Z(G)$. Then A is a hypercentral ascendant subgroup of G, and Lemma 2.1 yields that A is finite. Therefore also the centre $Z(G)$ of G is finite, and it follows from the same lemma that every abelian ascendant subgroup of G is finite. In particular, the Gruenberg radical K of G is periodic and satisfies the minimal condition on abelian ascendant subgroups. Then K is a Černikov group (see [7], Theorem E), and hence it is finite, so that also G is finite. This contradiction proves that the group G is abelian.

(b) Suppose first that G is locally (soluble-by-finite), so that without loss of generality it can be assumed that G is finitely generated, and hence soluble-by-finite. Let S be the largest soluble normal subgroup of G, and let N be any nilpotent normal subgroup of S. If A is a maximal abelian normal subgroup of N, then

$$N = N_N(A) = C_N(A) = A,$$

so that N is abelian. It follows that the Fitting subgroup F of S is abelian, and hence

$$S = N_S(F) = C_S(F) = F$$

is also abelian. Thus

$$G = N_G(S) = C_G(S),$$

so that S is the centre of G and $G/Z(G)$ is finite. In particular, the commutator subgroup G' of G is finite (see [8] Part 1, Theorem 4.12), and hence it is abelian (see [11]). Thus G is soluble, and so abelian. In the general case, it follows from the first part of the proof that the Gruenberg radical K of G is abelian and contains every locally (soluble-by-finite) ascendant subgroup of G. Then the factor group G/K does not contain locally (soluble-by-finite) non-trivial ascendant subgroups, and hence by hypothesis $G = K$ is an abelian group. □

It can be observed that statement (a) of Theorem A cannot be extended to radical groups (i.e. groups with an ascending series whose factors are locally nilpotent), even if the condition of having small automizer is imposed to every abelian ascendant subgroup. In fact, there exists an infinite locally finite p-group (p prime) with trivial Gruenberg radical (see [8] Part 2, p.29).

PROPOSITION 2.2. *Let G be a hyperabelian group such that $G = HC_G(H)$ for every normal subgroup H of G. Then G is abelian.*

Proof. Clearly every abelian normal subgroup of G is contained in the centre $Z(G)$, and since the hypotheses are inherited by homomorphic images, it follows that the group G is hypercentral. If A is a maximal abelian normal subgroup of G, then $C_G(A) = A$, and hence $G = A$ is abelian. □

3. Small automizers of non-abelian subgroups.

We begin this section characterizing locally nilpotent groups with the property $SANS$.

THEOREM 3.1. *Let G be a locally nilpotent non-abelian group. Then G is an $SANS$-group if and only if the factor group $G/Z(G)$ has order p^2 for some prime number p.*

Proof. Suppose first that G is an $SANS$-group, and let E be any finitely generated subgroup of G. Then E is residually finite, and all its finite homomorphic images are $SANS$-groups, so that they have nilpotency class at most 2 (see [2]). It follows that E has nilpotency class at most 2, and hence also G is a nilpotent group of class 2. Let A be a maximal abelian normal subgroup of G, and let H be any subgroup of G properly containing A. Then H is a non-abelian normal subgroup of G, and so $G = HC_G(H)$. On the other hand,

$$C_G(H) \leq C_G(A) = A,$$

so that $G = H$ and the factor group G/A must have prime order p. Let x be an element of the set $G \setminus A$, and consider a maximal abelian normal subgroup B of G containing x. The above argument shows that G/B has prime order q. Since $A \cap B$ is contained in $Z(G)$, it follows that $G/Z(G)$ is an abelian group of order pq. Therefore $p = q$ and $G/Z(G)$ has order p^2.

Conversely, suppose that the factor group $G/Z(G)$ has order p^2, where p is a prime, and let H be a non-abelian subgroup of G. Clearly G' has order p, so that $H' = G'$, and H is normal in G. Moreover $Z(G)$ is contained in $C_G(H)$, and

$$HC_G(H)/C_G(H) \cong H/Z(H)$$

has order p^2. Therefore $G = HC_G(H)$, and so G is an $SANS$-group.□

LEMMA 3.2. *Let G be a locally soluble non-abelian $SANS$-group. Then G contains an abelian normal subgroup of prime index. In particular, G is metabelian.*

Proof. Suppose that the group G is soluble, and let F be the Fitting subgroup of G. If F is not abelian, then $G = FC_G(F) = F$, and hence G is locally nilpotent. In this case, it follows from Theorem 3.1 that $G/Z(G)$ has order p^2, where p is a prime number, so that G contains an abelian normal subgroup of index p. Suppose now that F is abelian, and let H be any normal subgroup of G properly containing F. Then H is not abelian and $C_G(H) \leq C_G(F) = F$, so that $G = HC_G(H) = H$. Therefore G/F is a simple group, and hence has prime order. In the general case, it follows from the first part of the proof that every finitely generated subgroup of G is metabelian, so that G itself is metabelian. The lemma is proved. □

LEMMA 3.3. *Let G be a group, and let K be a subgroup of G such that $G = KZ(G)$. If K is an $SANS$-group, then also G is an $SANS$-group.*

Proof Consider first a non-abelian subgroup H of G containing $Z(G)$. Clearly

$$H = H \cap KZ(G) = Z(G)(H \cap K),$$

and $H \cap K$ is not abelian, so that

$$N_K(H \cap K) = (H \cap K)C_K(H \cap K).$$

Then

$$N_G(H) = N_G(H \cap K) = Z(G)N_K(H \cap K)$$

$$= Z(G)(H \cap K)C_K(H \cap K) = HC_G(H \cap K) = HC_G(H).$$

Suppose now that H is an arbitrary non-abelian subgroup of G. It follows from the first part of the proof that

$$N_G(HZ(G)) = HZ(G)C_G(HZ(G)) = HC_G(H).$$

As $N_G(H) \leq N_G(HZ(G))$, we obtain that $N_G(H) = HC_G(H)$, and G is an $SANS$-group. □

It was proved in [2] that, if $G = A \times B$ is a finite $SANS$-group and A is not abelian, then B must be abelian. The proof of this result actually works replacing the condition that G is finite with any property suitable to obtain an extension of the theorem of Zassenhaus. In particular, it follows from Theorem A that this is true in the case of locally finite groups.

Proof of Theorem B Suppose that G is an $SANS$-group, and assume first that it is also locally soluble, so that G contains an abelian normal subgroup A of prime index by Lemma 3.2. Obviously $A = Fit\,G$, and hence $G/Fit\,G$ has prime order p. Then $G = \neg P \ltimes K$, where K is an abelian normal p'-subgroup of G and P is any Sylow p-subgroup of G. Assume that P is not abelian, so that $P/Z(P)$ has order p^2 by Theorem 3.1, and hence there exists a maximal abelian subgroup B of P such that $V = KB \neq A$. As G is not nilpotent, the normal subgroup V of G is not abelian, so that $G = VC_G(V)$, and there exists an element x of $C_G(V)$ such that $G = KB\langle x\rangle$. Clearly $B\langle x\rangle$ is abelian, so that also $P \cong G/K$ is an abelian group. This contradiction proves that all Sylow subgroups of G are abelian.

Suppose now that G is not locally soluble, so that it contains a finite non-soluble subgroup E. Let $\mathcal{L}$ be the set of all finite subgroups of G containing E. If H is any element of $\mathcal{L}$, then H is a finite non-soluble $SANS$-group, so that $H = H' \times Z(H)$ and H' is a simple non-abelian group (see [2]). Consider the normal subgroup

$$K = \langle H' | H \in \mathcal{L}\rangle$$

of G. Then $\{H' | H \in \mathcal{L}\}$ is a local system of K consisting of simple non-abelian groups, so that also K is a simple non-abelian group.

Therefore $G = KC_G(K)$ and $K \cap C_G(K) = Z(K) = 1$, so that $G = K \times C_G(K)$, and since K is not abelian, the subgroup $A = C_G(K)$ must be abelian. It follows from results in [2] and the list given by Dickson of subgroups of projective special linear groups (see [5], Hauptsatz 2.8.27) that chains of length 2 of finite simple non-abelian $SANS$-groups do not exist. Therefore K is finite, and it satisfies the condition of statement (b) by [2].

Conversely, suppose first that the Fitting subgroup F of G has prime index p, and that all Sylow subgroups of G are abelian. Then $G = P \ltimes K$, where P is a Sylow p-subgroup and K is an abelian normal p'-subgroup of G. Let E be a finite subgroup of P such that $P = E(P \cap F)$. Then

$$K = C_K(E) \times [K, E]$$

(see for instance [9], 10.1.6), and so also

$$K = C_K(P) \times [K, P],$$

since $P \cap F$ is a subgroup of $Z(G)$. As $C_K(P)$ is contained in $Z(G)$, we obtain that

$$F = PK \cap F = K(P \cap F)$$

lies in $G'Z(G)$, and hence $F = G'Z(G)$. If H is any non-abelian subgroup of G, the normalizer $N_G(H)$ inherits the structure of G, and hence without loss of generality it can be assumed that H is normal in G. Since F is abelian, H is not contained in F and $G = F\langle x\rangle$ for some element x of H. Then $G' = [F, x]$ is contained in H, and so

$$G = F\langle x\rangle = G'Z(G)\langle x\rangle = HC_G(H).$$

Therefore G is an $SANS$-group. Assume now that $G = A \times K$, where A is abelian and $K \simeq PSL(2, p^f)$ is a simple group such that $(p^f - 1)/h$ is a prime where $h = (p - 1, 2)$. Then K is an $SANS$-group (see [2]), and hence also G is an $SANS$-group by Lemma 3.3. □

LEMMA 3.4. *Let G be a locally soluble $SANS$-group. Then the commutator subgroup G' of G is periodic.*

Proof. Since every subgroup of an $SANS$-group is likewise an $SANS$-group, it can be assumed without loss of generality that G is a finitely generated soluble non-abelian group. Then G is polycyclic, as by Lemma 3.2 it contains an abelian normal subgroup A of prime index p.

Clearly the torsion subgroup T of A is finite, and G/T is residually a finite p-group. Since nilpotent $SANS$-groups have class at most 2, it follows that G/T is nilpotent, and hence finite-by-abelian by Theorem 3.1. Therefore the commutator subgroup G' of G is finite. □

LEMMA 3.5. *Let G be a group with periodic commutator subgroup, and let H be a subgroup of finite index of G. If $H/Z(H)$ is periodic, then also the factor group $G/Z(G)$ is periodic.*

Proof. Obviously it can be assumed that the subgroup H is normal in G. Let X be a transversal to H in G, and let z be any element of $Z(H)$. As G' is periodic, there exists a positive integer n such that $[z,x]^n = 1$ for every element x of X. Then $[z^n, x] = 1$ for all $x \in X$, and hence z^n belongs to $Z(G)$. Therefore the factor group $G/Z(G)$ is periodic. □

LEMMA 3.6. *Let G be a periodic non-abelian group containing an abelian normal subgroup A of prime index. If all Sylow subgroups of G are abelian, then $A = G'Z(G)$.*

Proof. Clearly, A is the Fitting subgroup of G, and hence contains $G'Z(G)$. Let x be an element of G such that $G = A\langle x\rangle$, and let y be an element of A such that $[x,y]\neq 1$. If a is any element of A, the finite normal subgroup $N = \langle a, x^p, y\rangle^G$ of G is the Fitting subgroup of $N\langle x\rangle$, and hence

$$N = (N\langle x\rangle)'Z(N\langle x\rangle)$$

(see [5], Satz 6.14.7). Moreover, $Z(N\langle x\rangle)$ is contained in the centre of G, so that $N \leq G'Z(G)$. Thus $A = G'Z(G)$, and the lemma is proved. □

Proof of Theorem C Suppose first that G is an $SANS$-group, so that by Lemma 3.2 it contains an abelian normal subgroup of prime index. Moreover, the commutator subgroup G' of G is periodic by Lemma 3.4, and applying Lemma 3.5 we obtain that also $G/Z(G)$ is a periodic group. Let B be a maximal torsion-free subgroup of $Z(G)$, so that G/B is a periodic non-nilpotent group, and all Sylow subgroups of G/B are abelian by Theorem $B(a)$.

Conversely, suppose that G contains an abelian normal subgroup A of prime index and a central torsion-free subgroup B such that G/B is periodic and all its Sylow subgroups are abelian. Since G is not nilpotent, A/B is the Fitting subgroup of G/B. Moreover, the commutator subgroup of G is periodic (see [8] Part 1, Corollary to Theorem 4.12), so that $Z(G/B) = Z(G)/B$ and hence $G'Z(G) = A$ by Lemma 3.6. Let H be any non-abelian subgroup of G. As the hypotheses on G are inherited by all subgroups, it can be assumed without loss of generality that H is normal in G. Let x be an element of $H \backslash A$. Then $G = A\langle x\rangle$ and

$$G' = [A, x] \leq \langle x\rangle^G \leq H.$$

Therefore $G = \langle x, G', Z(G)\rangle = HC_G(H)$, and G is an $SANS$-group. □

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