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ON ASCENDING AND SUBNORMAL SUBGROUPS OF INFINITE FACTORIZED GROUPS

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We consider an almost hyper-Abelian group G of a finite Abelian sectional rank that is the product of two subgroups A and B. We prove that every subgroup H that belongs to the intersection $A \bigcap B$ and is ascending both in A and B is also an ascending subgroup in the group G. We also show that, in the general case, this statement is not true.

1. Introduction

Let a group G be the product $G = A\,B$ of two subgroups A and B and let H be a subgroup from the intersection $A \bigcap B$. It is obvious that if the subgroup H is normal in both subgroups A and B, then it is also normal in the group G. In addition, Maier [1] and Wielandt [2] proved that if the group G is finite and the subgroup H is subnormal both in A and B, then the subgroup H is also subnormal in the group G. The problem of the validity of the corresponding result in the case of an infinite group G remains open. For some special cases, for example, for the case where a commutant G' of the group G is nilpotent, Stonehewer [3] gave a positive answer to this question and announced in [3] that a similar conclusion is also valid if G is a solvable minimax group. The aim of the present paper is to extend these results to wider classes of infinite groups. In the first theorem, we consider ascending subgroups of certain factorized groups of finite 0-rank. Recall that a group G has a finite 0-rank if it has a finite subnormal series factors of which are either periodic or infinitely cyclic, and a subgroup H of G is called ascending in G if it is a term of a certain ascending subnormal series of the group G.

Theorem 1. *Let a group G of finite 0-rank be the product $G = A\,B$ of two subgroups A and B and let a subgroup H from the intersection $A \bigcap B$ be an ascending subgroup both in A and B. If the group G has an ascending series of normal subgroups factors of which are either finite or Abelian groups without torsion, then the subgroup H is ascending in the group G.*

A group G is said to have a finite Abelian sectional rank if there are no infinite Abelian sections of a simple exponential in it. A group is called hyper-Abelian if it has an ascending series of normal subgroups with Abelian factors and almost hyper-Abelian if it contains a hyper-Abelian subgroup of finite index. It is easy to see that every almost hyper-Abelian group of a finite Abelian sectional rank has a finite 0-rank and an ascending series of normal subgroups whose factors are either finite or Abelian groups without torsion. For this reason, the following assertion, which gives a positive answer to the second part of question 16 in [4], is a direct corollary of Theorem A.

Corollary 1. *Let an almost hyper-Abelian group G of a finite Abelian sectional rank be the product $G = A\,B$ of two subgroups A and B. If H is a subgroup of $A \bigcap B$ that is ascending in both subgroups A and B, then H is an ascending subgroup of the group G.*

Recall that the Hirsch–Plotkin radical of a group is called its maximal locally nilpotent normal subgroup.

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According to [5] (Theorem 2.31), this radical contains every ascending locally nilpotent subgroup of the group. Noting now that every locally nilpotent group containing an ascending series of normal subgroups with Abelian factors of finite ranks is hypercentral (see, e.g., the corollary of Theorem 6.38 in [5]) and, therefore, every subgroup of this group is ascending, we conclude that the following assertion is also a corollary of Theorem A:

Corollary 2. *Let a group G of finite 0-rank be the product $G = AB$ of two subgroups A and B and let H and K be the Hirsch–Plotkin radicals of the subgroups A and B, respectively. If the group G has an ascending series of normal subgroups each factor of which is either finite or an Abelian group without torsion, then the intersection $H \cap K$ is contained in the Hirsch–Plotkin radical of the group G.*

The group G is called a $\mathfrak{S}_1$-group if it has a finite Abelian sectional rank and the set of normal divisors of orders of elements of its periodic subgroups is finite. The construction of hyper-Abelian $\mathfrak{S}_1$-groups is described in [5] (Theorem 10.33). In particular, such groups are solvable. It is also obvious that every solvable minimax group is a $\mathfrak{S}_1$-group, and, thus, the statement below slightly improves the result announced by Stonehewer [3] and, in addition, gives a positive answer to the first part of question 16 in [4] for $\mathfrak{S}_1$-groups.

Theorem 2. *Let an almost solvable $\mathfrak{S}_1$-group G be the product $G = AB$ of two subgroups A and B. If H is a subgroup of $A \cap B$ subnormal in A and B, then H is subnormal in G.*

The last theorem shows that, in the case of locally finite groups, a result similar to Theorem A is not true. Before we formulate the corresponding result, recall that a subgroup H of a group G is called a system subgroup if it is a member of a subnormal system of the group G. Obviously, every ascending subgroup is a system subgroup.

Theorem 3. *There exists a countable locally finite group of the form $G = AB$ that is an extension of an Abelian group by a locally nilpotent group and has the following properties:*

(i) the intersection $A \cap B$ is finite;

(i) every finite nontrivial subgroup from $A \cap B$ is ascending both in A and B but it is not a system subgroup of the group G.

The notation and definitions used here are, mainly, conventional and given in [4, 6]. Note only that the terms "subnormal series" and "subnormal system" accepted in the Russian mathematical literature are the equivalent to the term "series" in [5, p. 10].

2. Proof of Theorem 1

The following lemma is an elementary result about ascending and subnormal subgroups:

Lemma 1. *Let G be a group, let K and L be its normal subgroups, and let H be a subgroup of the group G such that $G = HKL$ and H is an ascending (subnormal) subgroup in HK and HL. Then H is an ascending (subnormal) subgroup of the group G.*

Proof. Let

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_\alpha \lhd H_{\alpha+1} \lhd \ldots \lhd H_\tau = HK$$

be an ascending (finite) subnormal series of the subgroup HK. Then the subgroup $H_\alpha L$ is normal in $H_{\alpha+1}L$ for every ordinal $\alpha < \tau$. For this reason, the subgroup $HL = H_0L$ is ascending (subnormal) in $G = HKL$ and, therefore, H is an ascending (subnormal) subgroup of the group G. Lemma 1 is proved.

Let N be a normal subgroup of a group G that is the product $G = AB$ of two subgroups A and B. The factorizer of the subgroup N in the group G is the subgroup $X(N) = AN \cap BN$ which has a so-called triple factorization

$$X(N) = A^*B^* = A^*N = B^*N,$$

where $A^* = A \cap BN$ and $B^* = B \cap AN$ (see Lemma 1.1.4 in [4]). It is obvious that $A^* \cap B^* = A \cap B$. Therefore, triple factorizations play a key role in our reasoning as well as in many other questions concerning factorized groups.

Lemma 2. *Let a group G be the product $G = AB = AK = BK$ of two subgroups A and B and a finite normal subgroup K. If a subgroup H from $A \cap B$ is ascending in A and B, then H is an ascending subgroup in G.*

Proof. Since the centralizers $C_A(K)$ and $C_B(K)$ are normal subgroups in G and have finite indices A and B, respectively, we conclude that their product $C = C_A(K)\,C_B(K)$ is a normal subgroup of finite index in G. Consider a finite quotient group $\overline{G} = G/C$ and let an over-bar denote a homomorphic image of the corresponding subgroup in this quotient group. It is obvious that $\overline{G} = \overline{A}\,\overline{B}$ and $\overline{H}$ is a subgroup of the intersection $\overline{A} \cap \overline{B}$ that is subnormal both in $\overline{A}$ and $\overline{B}$. Therefore, by the Maier–Wielandt theorem (see [4], Theorem 7.5.7), the subgroup $\overline{H}$ is subnormal in $\overline{G}$ and, consequently, the subgroup HC is subnormal in G. On the other hand, it is clear that the subgroup H is an ascending subgroup in $HC_A(K)$ and $HC_B(K)$ and, hence, by Lemma 1, it is ascending in HC. Therefore, H is an ascending subgroup in G. Lemma 2 is proved.

Lemma 3. *Let a group G be the product $G = AB = AK = BK$ of two subgroups A and B and a periodic normal subgroup K that has an ascending G-invariant series with finite factors. If a subgroup H of $A \cap B$ is ascending in both subgroups A and B, then H is an ascending subgroup in G.*

Proof. Let

$$1 = K_0 \lhd K_1 \lhd \ldots \lhd K_\alpha \lhd K_{\alpha+1} \lhd \ldots \lhd K_\tau = K$$

be an ascending G-invariant series with finite factors. For every ordinal $\alpha < \tau$, we denote by X_α a factorizer of a subgroup K_α in the group $G = AB$. Then

$$X_\alpha = A_\alpha B_\alpha = A_\alpha K_\alpha = B_\alpha K_\alpha,$$

where $A_\alpha = A \cap BK_\alpha$ and $B_\alpha = B \cap AK_\alpha$. If $\alpha < \tau$, then, by using Lemma 2 for the quotient group $X_{\alpha+1}/K_\alpha$, we obtain that HK_α is an ascending subgroup in $X_{\alpha+1}$ and, thus, in $HK_{\alpha+1}$. Therefore, H is an ascending subgroup in HK. Since it is obvious that the subgroup HK is ascending in G, the same is also true for the subgroup H. Lemma 3 is proved.

Lemma 4. *Let a group G have a finite 0-rank and let it be representable as the product $G = AB = AK = BK$ of two subgroups A and B and an Abelian normal group without torsion K. If a subgroup H of $A \cap B$ is ascending both in A and B, then H is an ascending subgroup in G.*

Proof. Assume that $A \cap K = B \cap K = 1$ and let L be a proper G-invariant subgroup of K maximal with respect to the condition that the quotient group K/L is a group without torsion. The induction on 0-rank of a subgroup K allows us to assume that the lemma holds for the factorizer $X(L)$ of a subgroup L in the group $G = AB$, so that a subgroup H is ascending in $X(L)$ and, in particular, in HL. Therefore, it is sufficient to prove that the subgroup HL is ascending in G. By replacing the group G with the quotient group G/L, we can assume that G acts on K rationally and irreducibly. In particular, the quotient group $A/C_A(K)$ can be regarded as an irreducible linear group over the field of rational numbers. Since the group G has a finite 0-rank, by the Tits alternative [6], this quotient group is almost solvable. Therefore, if $C = C_A(K)\, C_B(K)$, then the quotient group $\overline{G} = G/C$ is also almost solvable and, obviously, satisfies the conditions of Lemma 4.

If $C_A(K) = C_B(K)$, then the quotient group $\overline{G}$ satisfies the assumptions made above and $C_{\overline{A}}(\overline{K}) = 1$, whence it follows that $C_{\overline{G}}(\overline{K}) = \overline{K}$. According to the Wilson result ([4], Lemma 4.1b), this is impossible. Therefore, $C_A(K) \neq C_B(K)$. Since $C_G(K) = C_A(K)K = C_B(K)K$, we have $C = C_A(K)(K \cap C) = C_B(K)(K \cap C)$ and, thus, $K \cap C$ is a nontrivial normal subgroup in G. It is clear that the quotient group $\tilde{G} = G/(K \cap C)$ is a product of the form

$$\tilde{G} = \tilde{A}\tilde{B} = \tilde{A}\tilde{K} = \tilde{B}\tilde{K}$$

and $\tilde{K}$ is its periodic Abelian normal subgroup of finite rank. Therefore, $\tilde{K}$ has an ascending G-invariant series with finite factors and, hence, by Lemma 3, the subgroup $\tilde{H}$ is ascending in $\tilde{G}$. This implies that $H(K \cap C)$ is an ascending subgroup in G. On the other hand, it is ascending in $HC_A(K)$ and $HC_B(K)$ and, hence, by virtue of Lemma 1, H is an ascending subgroup in HC. In particular, the subgroup H is ascending in $H(K \cap C)$ and, thus, in G.

Let us consider a more general case. Consider the normal subgroup

$$N = (A \cap K)(B \cap K)$$

of a group G and denote by T/N a subgroup consisting of all elements of finite order of the quotient group K/N. Then T/N has an ascending G-invariant series with finite factors. By using Lemma 3 for the factorizer V/N of the subgroup T/N in the quotient group $G/N = (AN/N)(BN/N)$, we establish that the subgroup HN is ascending in the subgroup V and, in particular, in the subgroup HT. Furthermore, by Lemma 1, the subgroup H is ascending in HN and, therefore, in HT. Since K/T is a subgroup without torsion of the quotient group G/T and

$$(AT/T) \cap (K/T) = (BT/T) \cap (K/T) = 1,$$

it follows from the first part of the proof that the subgroup HT is ascending in the group G. Therefore, H is an ascending subgroup in G. Lemma 4 is proved.

Proof of Theorem 1. Let

$$1 = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending series of normal subgroups of a group G whose factors are either finite or Abelian groups without torsion. For every ordinal $\alpha \leq \tau$, let X_α be a factorizer of a subgroup G_α in the group $G = AB$. Then

$$X_\alpha = A_\alpha B_\alpha = A_\alpha G_\alpha = B_\alpha G_\alpha,$$

where $A_\alpha = A \cap B G_\alpha$ and $B_\alpha = B \cap A G_\alpha$. If $\alpha < \tau$, the quotient group $X_{\alpha+1}/G_\alpha$ has a natural triple factorization in which the factor $G_{\alpha+1}/G_\alpha$ is either finite or an Abelian group without torsion. By using Lemmas 2 and 4 for this quotient group, we conclude that the subgroup HG_α is ascending in $X_{\alpha+1}$ and, in particular, in $HG_{\alpha+1}$. Therefore, H is an ascending subgroup in G. Theorem 1 is proved.

3. Proof of Theorems 2 and 3

The following lemma is a special case of Theorem 2:

Lemma 5. *Let an almost solvable $\mathfrak{S}_1$-group G be representable as the product $G = AB = AK = BK$ of two proper subgroups A and B and a periodic complete Abelian normal subgroup K whose proper G-invariant subgroups are finite. If a subgroup H of $A \cap B$ is subnormal in both subgroups A and B, then H is a subnormal subgroup of the group G.*

Proof. First, the centralizers $C_A(K)$ and $C_B(K)$ are normal subgroups of the group G. Then we assume that $C = C_A(K)\,C_B(K)$ and show that the quotient group is almost polycyclic G/C. Since by Lemma 1, the subgroup H is subnormal in HC and, by virtue of Theorem A, the subgroup HC is ascending in G, we conclude that the subgroup HC and, therefore, the subgroup H are subnormal in G.

Let N be a nilpotent normal subgroup of G containing a subgroup K. Then the mutual commutant $[K, N]$ is a proper complete G-invariant subgroup of K, so that $[K, N] = 1$ and, hence, $N \leq C_G(K)$. Therefore, the centralizer $C_G(K)$ contains the Fitting subgroup of the group G and, hence, the quotient group $G/C_G(K)$ is almost polycyclic (see [5], Theorem 10.33). But, in this case, the quotient groups $A/C_A(K)$ and $B/C_B(K)$ are also almost polycyclic. Since these groups are isomorphic to the subgroups AC/C and BC/C of the quotient group $G/C = (AC/C)(BC/C)$, according to the Lennox–Roseblade–Zaitsev theorem, this quotient group G/C is almost polycyclic (see [4], Theorem 4.4.2). Lemma 5 is proved.

Proof of Theorem 2. Since, by Theorem 1, a subgroup H is ascending in the group G, we can assume that the group G is infinite and, therefore, its Fitting subgroup F is also infinite.

Assume that the group G does not contain nontrivial periodic normal subgroups. Then the subgroup F is nilpotent and the quotient group G/F is almost polycyclic (see [5], Theorem 10.33). Let K be the center of the subgroup F. Since F is a subgroup without torsion, the quotient group F/K is also without torsion (see [5], Theorem 2.25). This means that the quotient group $G/K = (AK/K)(BK/K)$ is a $\mathfrak{S}_1$-group. The induction on the 0-rank of the group G allows us to assume that Theorem 2 is valid for this quotient group, so that, according to the induction hypothesis, the subgroup HK is subnormal in G. Therefore, it is sufficient to show that the subgroup H is subnormal in a factorizer X of the subgroup K in the group G. The factorizer has the triple factorization

$$X = A^*B^* = A^*K = B^*K,$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. It is clear that $A^* \cap F \leq C_{A^*}(K)$ in $B^* \cap F \leq C_{B^*}(K)$, so that the quotient groups $A^*/C_{A^*}(K)$ and $B^*/C_{B^*}(K)$ are almost polycyclic. The subgroup $C = C_{A^*}(K)\,C_{B^*}(K)$ is normal in X and the quotient group X/C is the product of two almost polycyclic subgroups A^*/C and B^*/C. For this reason, by the Lennox–Roseblade–Zaitsev theorem, this quotient group is also almost polycyclic and, hence, the subgroup HC is subnormal in X. On the other hand, by Lemma 1, the subgroup H is subnormal in HC. Therefore, the subgroup H is subnormal in X and, hence, in G.

Now consider a more general case and denote by T the maximal periodic normal subgroup of a group G. The subgroup T is a Chernikov group [5, p. 139] and, therefore, it has a complete Abelian subgroup J that is unique and, hence normal in G, has a finite index in T, and is the direct product of finitely many quasicyclic groups. Since the quotient group $G/T = (AT/T)(BT/T)$ does not have nontrivial periodic normal subgroups, it follows from the first part of the proof that the subgroup HT is subnormal in G. In addition, the subgroup HJ, which is ascending in G, has a finite index in HT and, hence, HJ is subnormal in HT and, thus, in G. Therefore, to complete the proof, it is sufficient to show that the subgroup H is subnormal in HJ. By replacing the group G by a factorizer of the subgroup J in G, we can assume that the group G has the triple factorization

$$G = AB = AJ = BJ.$$

Let L be a subgroup maximal among proper complete G-invariant subgroups of J. The factorization of the subgroup L into the direct product of quasicyclic groups contains a smaller number of quasicyclic factors than the corresponding factorization of the subgroup J. By using the induction on this number, we conclude that the theorem is valid for the factorizer $X(L)$ of the subgroup L in the group $G = AB$. By virtue of this assumption, the subgroup H is subnormal in HL and it remains to show that the subgroup HL is subnormal in G. By replacing the group G with the quotient group G/L, we can assume that $L = 1$, i.e., that the subgroup J has no infinite proper G-invariant subgroups. But then, by Lemma 5, the subgroup H is subnormal in G. Theorem 2 is proved.

Proof of Theorem 3. According to [7, Corollary 1], for every prime number p, there exists a countable local finite group G that has the following factorization:

$$G = AB = AK = BK,$$

where A and B are p-subgroups and K is an Abelian normal subgroup of a simple exponential $q \neq p$. It is obvious that $C_A(K)$ is a normal subgroup in G. For this reason, passing to the quotient group $G/C_A(K)$, we can assume that $C_A(K) = 1$, which means that K is the Hirsch–Plotkin radical of the group G. Since A and B are countable locally nilpotent groups, every finite subgroup E of the intersection $A \cap B$ is ascending both in A and B. Assume that E is a system subgroup of the group G. Then the subgroup E is subnormal in every finite subgroup of G that contains this subgroup. By virtue of this fact, this subgroup is contained in K (see [8], Theorem 5.2.3.3). But then $E = 1$ and, hence, every finite nontrivial subgroup from the intersection $A \cap B$ is not a system subgroup of G. Let us now show that the intersection $A \cap B$ is infinite and, thus, complete the proof of the theorem.

Let A_0 be an arbitrary finite subgroup of A and let $B_0 = B \cap KA_0$. Then B_0 is a finite subgroup in B and $KA_0 = KB_0$ because A_0 and B_0 are finite Sylow p-subgroups of the group KA_0. Therefore, for an element x from K, we have $x^{-1}A_0x = B_0$. By taking into account that $x = ab$ for some elements a from A and b from B, we obtain that the subgroup $a^{-1}A_0a = b^{-1}B_0b$ is contained in $A \cap B$. Since the subgroup A is finite, as A_0, we can take a finite subgroup of arbitrarily large order and, therefore, the intersection $A \cap B$ is infinite. Theorem 3 is proved.

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