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Groups with finitely many normalizers of non-periodic subgroups

Abstract. A theorem of Polovickiĭ states that any group with finitely many normalizers of subgroups is finite over its centre. Here we prove that the centre of a non-periodic group G has finite index if and only if G has finitely many normalizers of non-periodic subgroups.

Keywords Normalizer subgroup · Central-by-finite group

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1 Introduction

A famous theorem of B.H. Neumann [4] states that a group is finite over its centre if and only if all its conjugacy classes of subgroups are finite. Since a subgroup X of a group G has finitely many conjugates if and only if its normalizer $N_G(X)$ has finite index in G, the above result suggests that the behaviour

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of normalizers of subgroups has a strong influence on the structure of groups. In fact, Y.D. Polovickiĭ [5] proved that a group is central-by-finite if and only if it has only finitely many normalizers of (abelian) subgroups. In recent years, groups with finitely many normalizers of subgroups with a given property θ have been studied for several possible choices of θ (the survey paper [1] can be used as a reference on this subject). The consideration of the infinite dihedral group shows that there exist non-periodic groups with trivial centre in which every non-periodic subgroup has only finitely many conjugates, so that Neumann's theorem cannot be generalized to the case of groups in which all non-periodic subgroups lie in finite conjugacy classes. On the other hand, the aim of this article is to prove that Polovickiĭ's result can be extended in this direction.

Theorem 1 *Let G be a non-periodic group with finitely many normalizers of non-periodic subgroups. Then the factor group $G/Z(G)$ is finite.*

Observe that our result cannot be improved assuming that the group has only finitely many normalizers of abelian non-periodic subgroups, since all abelian non-periodic subgroups of the infinite dihedral group are normal. Note also that Neumann's theorem has been sharpened by I.D. MacDonald [3], who proved that if in a group G no subgroup has more than k conjugates, then the index of the centre of G is bounded by a function of k; of course, it would be of interest to know whether our theorem can be improved in a similar way. Unfortunately, we have not been able to solve this problem, and we leave it here as an open question.

Most of our notation is standard and can be found in [6].

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2 Proof of the theorem

In the study of groups with finitely many normalizers of non-periodic subgroups, the first case to be considered is of course that of groups in which every subgroup is either periodic or normal. This is solved by the following easy result that was already proved in [2].

Lemma 1 *Let G be a non-periodic group whose non-periodic subgroups are normal. Then G is abelian.*

Proof Let a be any element of infinite order of G. As the hypothesis on non-periodic subgroups is inherited by subgroups and homomorphic images, the group G has no infinite dihedral sections and so its normal subgroup $\langle a \rangle$ is contained in $Z(G)$. Moreover $G/\langle a \rangle$ is a Dedekind group, so that G is nilpotent and hence it is generated by its elements of infinite order. Therefore G is abelian. □

Our next lemma shows that any group with only finitely many normalizers of non-periodic subgroups contains a subgroup of finite index in which all non-periodic subgroups are subnormal with defect at most 2.

Lemma 2 *Let G be a group with only finitely many normalizers of non-periodic subgroups. Then G contains a characteristic subgroup M of finite index such that $N_M(X)$ is normal in M for each non-periodic subgroup X of M.*

Proof If X is any non-periodic subgroup of G, its normalizer $N_G(X)$ obviously has finitely many images under automorphisms of G; in particular, the subgroup $N_G(X)$ has finitely many conjugates in G and so the index $|G : N_G(N_G(X))|$ is finite. It follows that also the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in \mathrm{Aut}(G)} N_G(N_G(X))^\alpha$$

has finite index in G. Let $\mathscr{H}$ be the set of all non-periodic subgroups of G. If X and Y are elements of $\mathscr{H}$ such that $N_G(X) = N_G(Y)$, then $M(X) = M(Y)$, and hence also

$$M = \bigcap_{X \in \mathscr{H}} M(X)$$

is a characteristic subgroup of finite index of G. Let X be any non-periodic subgroup of M. Then

$$M \leq M(X) \leq N_G(N_G(X)),$$

and so the normalizer $N_M(X) = N_G(X) \cap M$ is a normal subgroup of M. $\qquad\square$

Our next result shows that non-periodic groups with finitely many normalizers of non-periodic subgroups locally satisfy the maximal condition on subgroups. If G is any group, recall that the *Baer radical* of G is the subgroup generated by all cyclic subnormal subgroups of G; it is well known that the Baer radical is locally nilpotent.

Corollary 3 *Let G be a finitely generated non-periodic group with only finitely many normalizers of non-periodic subgroups. Then every subgroup of G is finitely generated.*

Proof It follows from Lemma 2 that G contains a normal subgroup M of finite index such that all non-periodic subgroups of M are subnormal. Then all elements of infinite order of M belong to the Baer radical B of M, and so the factor group M/B is periodic. On the other hand, M is finitely generated and M/B has finitely many normalizers of subgroups, so that M/B is finite by Polovickiĭ's theorem. Thus B is finitely generated and hence it is nilpotent. Therefore G is nilpotent-by-finite and so all its subgroups are finitely generated. $\qquad\square$

Lemma 4 *Let G be a group with only finitely many normalizers of non-periodic subgroups, and let A be a torsion-free abelian subgroup of G. Then $N_G(A) = C_G(A)$.*

Proof Assume for a contradiction that the statement is false, and choose a counterexample G with a minimal number k of normalizers of non-periodic subgroups. As the normalizer $N_G(A)$ is likewise a counterexample, it cannot have fewer than k normalizers of non-periodic subgroups, so that $N_G(A) = G$ and A is a normal subgroup of G.

Let x be an element of G such that $[A,x] \neq \{1\}$, and suppose first that $A \cap \langle x \rangle = \{1\}$, so that in particular the subgroup $\langle x \rangle$ is not normal in G. Consider an element a of A such that $[a,x] \neq 1$, and put $B = \langle a,x \rangle \cap A$. Then B is finitely generated by Corollary 3, and so there is a prime number p such that $[B^{p^n}, x] \neq \{1\}$ for each non-negative integer n. As

$$\bigcap_{n \geq 0} \langle B^{p^n}, x \rangle = \langle x \rangle,$$

there exists $m \geq 0$ such that the subgroup $\langle B^{p^m}, x \rangle$ is not normal in G. It follows that $\langle B^{p^m}, x \rangle$ has fewer than k normalizers of non-periodic subgroups, and hence the minimal choice of G yields that B^{p^m} is contained in the centre of $\langle B^{p^m}, x \rangle$, a contradiction.

Therefore $A \cap \langle x \rangle \neq \{1\}$, so that x has infinite order and $A \cap \langle x \rangle = \langle x^r \rangle$ for some $r > 1$. Let $T/\langle x^r \rangle$ be the subgroup consisting of all elements of finite order of $A/\langle x^r \rangle$. Then T is normal in $\langle A,x \rangle$ and $[A,x] \leq T$ by the previous case. As $\langle T,x \rangle/\langle x^r \rangle$ is locally finite and x^r belongs to $Z(\langle A,x \rangle)$, it follows from the well known Schur's theorem that the commutator subgroup $\langle T,x \rangle'$ is locally finite; in particular $[T,x]$ is locally finite. so that $[T,x] = \{1\}$ and hence $[A,x,x] = \{1\}$. Thus

$$[A,x]^r = [A,x^r] = \{1\}$$

and so also $[A,x] = \{1\}$. This last contradiction completes the proof of the lemma. $\square$

Lemma 5 *Let G be a soluble-by-finite group in which every torsion-free abelian normal section is central. Then the commutator subgroup G' of G is periodic.*

Proof Assume for a contradiction that the statement is false, and choose a counterexample G whose largest soluble normal subgroup S has minimal derived length n. If T is the largest periodic normal subgroup of G, the factor group G/T is likewise a minimal counterexample, and hence we may suppose without loss of generality that G has no periodic non-trivial normal subgroups. Let K be the smallest non-trivial term of the derived series of S, and let A be

a maximal abelian normal subgroup of G containing K. Clearly, A is torsion-free abelian, and so it is contained in the centre of G; it follows from Schur's theorem that the index $|G : A|$ is infinite, and in particular $n > 1$. Let U/A be the largest periodic subgroup of G/A; then $U/Z(U)$ is locally finite, so that U' is locally finite and hence $U' = \{1\}$. Therefore $U = A$ and G/A has no periodic non-trivial normal subgroups. On the other hand, the factor group G/A has periodic commutator subgroup by the minimal choice of G, so that G/A is abelian and G is nilpotent. Then $C_G(A) = A$ and so $G = A$ is abelian. This contradiction proves the lemma. □

Since the property of having finitely many normalizers of non-periodic subgroups is inherited by homomorphic images, the following statement is a direct consequence of the previous results.

Corollary 6 *Let G be a group with finitely many normalizers of non-periodic subgroups. Then the commutator subgroup G' of G is periodic.*

We are now in a position to prove our theorem. In the proof we will use the elementary fact that if G is any abelian-by-finite group with finite commutator subgroup, then the factor group $G/Z(G)$ is finite.

PROOF OF THE THEOREM − Assume that the statement is false, and choose a counterexample G such that the set

$$\{N_G(X_1), \ldots, N_G(X_k)\}$$

of all proper normalizers of non-periodic subgroups has minimal size k. If $k = 0$, every non-periodic subgroup of G is normal and hence G is abelian by Lemma 1. Suppose that $k \geq 1$, and assume first that

$$G = N_G(X_1) \cup \ldots \cup N_G(X_k).$$

It follows from a well known result of B.H. Neumann that

$$G = N_G(X_{i_1}) \cup \ldots \cup N_G(X_{i_r})$$

for a suitable subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$ such that each $N_G(X_{i_j})$ has finite index in G (see [6] Part 1, Lemma 4.17). Then for each $j = 1, \ldots, r$ the subgroup $N_G(X_{i_j})$ is non-periodic and has fewer than k proper normalizers of non-periodic subgroups, so that every $N_G(X_{i_j})$ is central-by-finite by the minimal choice of G. Therefore G is abelian-by-finite. Moreover, the commutator subgroup $N_G(X_{i_j})'$ of $N_G(X_{i_j})$ is finite and has finitely many conjugates, so that the normal closure

$$N = \langle N_G(X_{i_1})', \ldots, N_G(X_{i_r})' \rangle^G$$

is finite by Dietzmann's Lemma. The factor group G/N is covered by finitely many abelian subgroups, so that it is central-by-finite (see [6] Part 1, Theorem

4.16) and hence also finite-by-abelian. Thus the group G itself is finite-by-abelian and so even central-by-finite. This contradiction shows that the set

$$N_G(X_1) \cup \ldots \cup N_G(X_k)$$

is properly contained in G.

Let g be an element of

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)),$$

so that every non-periodic subgroup of G which is normalized by g must be normal in G. Let a be an element of infinite order of G. Then the subgroup $\langle a, g \rangle$ is normal in G and $G/\langle a, g \rangle$ is a Dedekind group. In particular, it follows from Corollary 3 that the commutator subgroup G' is finitely generated. On the other hand, G' is periodic (and so even locally finite) by Corollary 6, and hence G' is finite. Thus the centralizer $C_G(g)$ has finite index in G, so that it is non-periodic; since all non-periodic subgroups of $C_G(g)$ are normal, it follows from Lemma 1 that $C_G(g)$ is abelian. Therefore the group G is abelian-by-finite and hence $G/Z(G)$ is finite. This last contradiction completes the proof of the theorem. $\square$

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