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A NOTE ON GROUPS WITH FINITELY MANY MAXIMAL NORMALIZERS

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Groups with finitely many maximal normalizers of non-abelian subgroups are studied in this paper. It is proved in particular that (generalized) soluble groups with such property have finite commutator subgroup. This result is an extension of a well-known theorem by Romalis and Sesekin on groups in which every non-abelian subgroup is normal.

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1. Introduction

In a famous paper of 1955, B. H. Neumann [4] proved that each subgroup of a group G has finitely many conjugates if and only if its centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to conjugacy classes of abelian subgroups (see [2]). Thus central-by-finite groups are precisely those groups in which the normalizers of abelian subgroups have finite index, and this fact suggests that the behaviour of normalizers has a strong influence on the structure of the group. In fact, it was shown later by Y. D. Polovickiĭ [5] that a group has finitely many normalizers of abelian subgroups if and only if it is central-by-finite.

A group G is called *metahamiltonian* if all its non-abelian subgroups are normal. Groups with this property were introduced and investigated by G. M. Romalis and N. F. Sesekin ([8],[9],[10]), who proved in particular that (generalized) soluble metahamiltonian groups have finite commutator subgroup. This result has recently been extended to the case of groups with finitely many normalizers of non-abelian subgroups (see [1]). The aim of this paper is to prove a similar theorem for groups with weaker restrictions on the set of all normalizers of non-abelian subgroups.

Let θ be a property pertaining to subgroups of groups. A group G is said to have *finitely many maximal normalizers* of θ-subgroups if there exists a finite set $\{X_1, \ldots, X_k\}$ of non-normal θ-subgroups of G such that for each non-normal θ-subgroup H of G the normalizer $N_G(H)$ is contained in $N_G(X_i)$ for some i. The structure of groups with few maximal normalizers of subgroups has been investigated in [11], where the authors prove, among other results, that if a group G has at most two maximal normalizers of subgroups, then G' is finite. Here we prove the following result.

Theorem A *Let G be a $\mathcal{W}$-group having finitely many maximal normalizers of non-abelian subgroups. Then the commutator subgroup G' of G is finite.*

Recall that a group G is said to be a *$\mathcal{W}$-group* if every finitely generated non-nilpotent subgroup of G has a finite non-nilpotent homomorphic image. The assumption that the group has the property $\mathcal{W}$ is required in the above statement in order to avoid Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and other similar pathological examples within the universe of metahamiltonian groups. Of course, the result by Romalis and Sesekin is a special case of our Theorem A.

The main difficulty in the proof of Theorem A depends on the fact that the property of having finitely many maximal normalizers of (non-abelian) subgroups is neither inherited by subgroups nor by homomorphic images. Thus we will obtain our result as a corollary of a more general statement. If θ is a subgroup property, we shall say that a set $\mathfrak{X}$ of proper subgroups of a group G is a *normalizing set* for θ-subgroups if for each non-normal θ-subgroup H of G there exists an element X of $\mathfrak{X}$ such that $N_G(H) \leq X$. Clearly, the set of all proper normalizers of θ-subgroups of G is a normalizing set for θ-subgroups; in particular, if a group G has finitely many maximal normalizers of θ-subgroups, then it has a finite normalizing set for θ-subgroups. Therefore Theorem A is a consequence of the following result.

Theorem B *Let G be a $\mathcal{W}$-group having a finite normalizing set for non-abelian subgroups. Then the commutator subgroup G' of G is finite.*

Most of our notation is standard and can be found in [6].

2. Proof of Theorem B

Our first elementary lemma shows that the hypotheses of Theorem B are inherited by homomorphic images.

Lemma 1. *Let G be a group and let $\mathfrak{X}$ be a normalizing set for non-abelian subgroups of G. If N is a normal subgroup of G, the set*

$$\mathfrak{X}_{G/N} = \{X/N \mid N \leq X \in \mathfrak{X}\}$$

is a normalizing set for non-abelian subgroups of G/N.

Proof. If H/N is any subgroup of G/N which is neither abelian nor normal, there exists an element X of $\mathfrak{X}$ such that $N_G(H) \leq X$. Then N is contained in X and hence X/N belongs to $\mathfrak{X}_{G/N}$. Therefore $\mathfrak{X}_{G/N}$ is a normalizing set for non-abelian subgroups of G/N. □

Although the property of having a finite normalizing set for non-abelian subgroups is not inherited by subgroups, we note here the following obvious property.

Lemma 2. *Let G be a group and let $\mathfrak{X}$ be a normalizing set for non-abelian subgroups of G. If K is a subgroup of G which is contained in no elements of $\mathfrak{X}$, then the set*

$$\mathfrak{X}_K = \{X \cap K \mid X \in \mathfrak{X}\}$$

is a normalizing set for non-abelian subgroups of K.

If G is a group, we shall denote by $\Lambda(G)$ the intersection of all maximal non-normal subgroups of G (with the convention that $\Lambda(G) = G$ if all maximal subgroups of G are normal). Then $\Lambda(G)$ is a characteristic subgroup of G containing the Frattini subgroup $\Phi(G)$, and the factor group $\Lambda(G)/\Phi(G)$ is abelian. Moreover, it follows directly from Frattini's argument that $\Lambda(G)$ is nilpotent for any finite group G.

Lemma 3. *Let G be a $\mathcal{W}$-group having a finite normalizing set for non-abelian subgroups. Then G is locally (nilpotent-by-finite).*

Proof. Let

$$\mathfrak{X} = \{X_1, \dots, X_k\}$$

be a finite normalizing set for non-abelian subgroups of G, and consider an element $y_i \in G \setminus X_i$ for each $i = 1, \dots, k$. Let E be any finitely generated subgroup of G and put $V = \langle E, y_1, \dots, y_k \rangle$. Then

$$\mathfrak{X}_V = \{X_1 \cap V, \dots, X_k \cap V\}$$

is a normalizing set for non-abelian subgroups of V by Lemma 2. Assume for a contradiction that V is not nilpotent-by-finite. Since G satisfies the property $\mathcal{W}$, we have in particular that V contains a normal subgroup N such that V/N is a finite non-nilpotent group. If V contains a maximal subgroup M which is abelian, the index $|V : M|$ must be infinite, so that $V = MN$ and V/N is abelian; this contradiction shows that no maximal subgroups of V are abelian. It follows that V contains only finitely many maximal subgroups which are not normal, and of course all such subgroups have finite index in V. Therefore the group $V/\Lambda(V)$ is finite and so $\Lambda(V)$ is finitely generated. Let H be any normal subgroup of finite index of $\Lambda(V)$; the core $K = H_V$ of H in V likewise has finite index in V, and hence $\Lambda(V)/K = \Lambda(V/K)$ is nilpotent. It follows that all finite homomorphic images of $\Lambda(V)$ are nilpotent, so that $\Lambda(V)$ itself is nilpotent, as G is a $\mathcal{W}$-group. This last contradiction proves that V is nilpotent-by-finite, and so G is locally (nilpotent-by-finite). □

For our purposes we also need the following result proved by J. C. Lennox (see [3]).

Lemma 4. *Let G be a finitely generated soluble-by-finite group. If the Frattini factor group $G/\Phi(G)$ is finite-by-nilpotent, then G is finite-by-nilpotent.*

We are now in a position to prove the main result of the paper.

Proof of Theorem B – Assume for a contradiction that G has infinite commutator subgroup. Let

$$\mathfrak{X} = \{X_1, \dots, X_k\}$$

be a finite normalizing set for non-abelian subgroups of G, and let y_i be an element of $G \setminus X_i$ for each $i = 1, \dots, k$. Consider a finitely generated non-abelian subgroup E of G containing $\langle y_1, \dots, y_k \rangle$. Then E is normal in G and the factor group G/E is a Dedekind group; in particular, $\langle G', E \rangle$ is finitely generated and hence it is nilpotent-by-finite by Lemma 3. Thus also G' is finitely generated and so there exists a finitely generated subgroup U of G such that $E \leq U$ and $U' = G'$. As

$$\{X_1 \cap U, \dots, X_k \cap U\}$$

is a normalizing set for non-abelian subgroups of U by Lemma 2, the group U is likewise a counterexample. Therefore we may replace G by U, and hence assume without loss of generality that G is finitely generated. In particular, G is nilpotent-by-finite and hence all its maximal subgroups have finite index. On the other hand, the group $G/Z(G)$ is infinite by Schur's theorem and so G contains at most one abelian maximal subgroup. Since any non-abelian maximal subgroup of G is either normal or belongs to $\mathfrak{X}$, it follows that G contains only finitely many maximal subgroups which are not normal. Thus the subgroup $\Lambda(G)$ has finite index in G, and so the Frattini factor group $G/\Phi(G)$ is finite-by-abelian. Therefore the group G is finite-by-nilpotent by Lemma 4.

As G satisfies the maximal condition on subgroups, we may consider a normal subgroup N of G which is maximal with respect to the condition that G/N has infinite commutator subgroup. Replacing now G by the counterexample G/N, we may also suppose that all proper homomorphic images of G are finite-by-abelian. Thus G is a torsion-free nilpotent group of class 2 and its centre $Z(G)$ is cyclic (see [7]); in particular, $G' = \langle c \rangle$ is infinite cyclic and

$$G/Z(G) = \langle g_1 Z(G) \rangle \times \ldots \times \langle g_r Z(G) \rangle,$$

where each $\langle g_i Z(G) \rangle$ is infinite cyclic. Let p be any prime number. Since

$$G = \langle g_1, \ldots, g_r \rangle Z(G)$$

and the group $G/\langle c^p \rangle$ is not abelian, there is a positive integer $i(p) \leq r$ such that the subgroup $\langle g_{i(p)}, c^p \rangle$ is not normal in G. Consider an index $j \leq r$ such that $[g_{i(p)}, g_j] \neq 1$. As the group $\langle g_{i(p)}, g_j^p, c^p \rangle / \langle c^p \rangle$ is free abelian of rank 2, there exists a prime number q such that the subgroup

$$H_p = \langle g_{i(p)}, g_j^{pq}, c^p \rangle$$

is not normal in G. Clearly, H_p is not abelian and hence the normalizer of H_p is contained in some element of $\mathfrak{X}$. It follows that there exist an element X of $\mathfrak{X}$ and an infinite set π of prime numbers such that $N_G(H_p) \leq X$ for all $p \in \pi$. On the other hand, as the commutator subgroup of $G/\langle c^p \rangle$ has order p, we have

$$G^p \leq N_G(H_p) \leq X$$

for all primes p, which is impossible since X is a proper subgroup of G. This last contradiction completes the proof of the theorem. □

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