

Locally Finite Products of Totally Permutable Nilpotent Groups

Maria De Falco Francesco de Giovanni[†]
Carmela Musella

*Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II
Complesso Universitario Monte S. Angelo, Via Cintia I - 80126, Napoli, Italy*
E-mail: mdefalco@unina.it degiovan@unina.it
cmusella@unina.it

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Abstract. A group $G = AB$ is said to be totally factorized by its subgroups A and B if $XY = YX$ for all subgroups X of A and Y of B . It is known that any finite group totally factorized by supersoluble subgroups is supersoluble, and that a finite group totally factorized by nilpotent subgroups is abelian-by-nilpotent. This latter result is extended here to certain classes of infinite groups.

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1 Introduction

Let the group $G = AB$ be the product of its subgroups A and B . In a famous article of 1955, Itô proved that if the factors A and B are abelian, then G is metabelian (see [1, Theorem 2.1.1]). A few years later, it was shown by Kegel and Wielandt that any finite group factorized by two nilpotent subgroups is soluble (see [1, Theorem 2.4.3]). On the other hand, every group of order $p^m q^n$ (where p and q are prime numbers) is the product of two nilpotent subgroups, and hence very little can be said in general about the structure of such factorized groups. Thus, it is natural to consider products of (generalized) nilpotent groups with additional requirements on the behaviour of factors.

Two subgroups A and B of a group G are said to be *mutually permutable* if $AY = YA$ and $XB = BX$ for all subgroups X of A and Y of B . Moreover, A and B are said to be *totally permutable* if $XY = YX$ for all $X \leq A$ and $Y \leq B$; if the

[†]Corresponding author.

group $G = AB$ is the product of its totally permutable subgroups A and B , we will also say that G is *totally factorized* by A and B . Factorizations of these types have been introduced for finite groups by Asaad and Shaalan [2]. Although a finite group factorized by two supersoluble normal subgroups need not be supersoluble (this being the case only when the commutator subgroup of the group is nilpotent), Asaad and Shaalan were able to prove that any finite group $G = AB$ totally factorized by two supersoluble subgroups A and B is supersoluble, and the same statement remains true also when the factors A and B are just mutually permutable, provided that at least one of them is nilpotent. Recently many authors have investigated the behaviour of totally factorized groups, especially in the case of finite groups. In particular, Ballester Bolinches, Cossey and Esteban Romero [4] proved that if the finite group $G = AB$ is the product of its totally permutable nilpotent subgroups A and B , then G is abelian-by-nilpotent, i.e., the last term of the lower central series of G is abelian. Although the quoted theorems of Asaad and Shaalan can be extended to locally finite groups by means of local arguments, this is not the case for the latter result and the aim of this paper is to generalize it to certain classes of infinite groups, in particular to locally finite groups with finite abelian section rank.

Recall that a group has *finite abelian section rank* if it has no infinite abelian sections of prime exponent. Thus, every locally finite primary group with finite abelian section rank satisfies the minimal condition on abelian subgroups and hence is a Černikov group (see [8, Part 1, Theorem 3.32]); it follows that any locally finite group with finite abelian section rank satisfies the condition $\text{min-}p$ for all prime numbers p .

Let \mathfrak{X} be a class of groups. If G is any group, the intersection of all normal subgroups N of G such that G/N is an \mathfrak{X} -group is called the \mathfrak{X} -residual of G ; moreover, the group G is said to be *residually \mathfrak{X}* if its \mathfrak{X} -residual is trivial. In particular, the *finite residual* of G is the intersection of all (normal) subgroups of finite index of G , and the nilpotent residual of G is the intersection

$$\gamma_\omega(G) = \bigcap_{n \in \mathbb{N}} \gamma_n(G),$$

where for each positive integer n , the subgroup $\gamma_n(G)$ is the n -th term of the lower central series of G . It is clear that in general residually \mathfrak{X} groups need not belong to \mathfrak{X} ; for instance, it is well known that every free group is residually (finite and nilpotent).

Our first theorem shows that locally finite groups with finite abelian section rank totally factorized by residually nilpotent subgroups are residually (abelian-by-nilpotent).

Theorem A. *Let the locally finite group $G = AB$ be the product of its totally permutable residually nilpotent subgroups A and B . If G has finite abelian section rank, then*

$$\bigcap_{n \in \mathbb{N}} \gamma_n(G)' = \{1\}.$$

In particular, the nilpotent residual of G is abelian.

It is easy to see that in the above statement, the assumption that G has finite abelian section rank can be replaced by the weaker condition that both factors A and B have such property. Theorem A is a consequence of another generalization of the quoted result by Ballester Bolinches, Cossey and Esteban Romero. We will say that a group G is *centrally \mathfrak{F} -perfect* if its finite residual J is contained in $Z(G)$ and the index $|G : J|$ is finite. Thus, the finite residual of any centrally \mathfrak{F} -perfect group is divisible abelian; moreover, locally nilpotent centrally \mathfrak{F} -perfect groups are clearly nilpotent and it is known that any nilpotent group satisfying the minimal condition on subgroups is centrally \mathfrak{F} -perfect (see [8, Part 1, Theorem 3.14]).

Theorem B. *Let the periodic group $G = AB$ be the product of its totally permutable locally nilpotent subgroups A and B . If the Sylow subgroups of A and B are centrally \mathfrak{F} -perfect, then*

$$\bigcap_{n \in \mathbb{N}} \gamma_n(G)' = \{1\},$$

and in particular, the nilpotent residual of G is abelian.

Most of our notation is standard and can be found in [8]; for properties concerning factorized groups, we refer to the monograph [1].

2 Proof of the Theorems

Let the group $G = AB$ be the product of its totally permutable subgroups A and B . Then $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$ for all elements $x \in A$ and $y \in B$, and the subgroup $\langle x, y \rangle$ is metabelian by Itô's theorem. On the other hand, it is well known that every soluble product of two periodic groups is periodic (see [1, Theorem 3.2.6]), so it follows that arbitrary groups factorized by periodic totally permutable subgroups are likewise periodic. Moreover, if the totally permutable factors A and B are π -groups, then G is also a π -group (where π is any set of prime numbers); note also that if A and B are locally finite, then G itself is locally finite since every finite subset of G is contained in a product of the form $XY = YX$, where X and Y are suitable finite subgroups of A and B , respectively. Finally, it is easy to show that products of totally factorized Černikov groups are likewise Černikov groups.

The following lemma has been proved by Jabara [7] and is relevant in the study of totally factorized groups. As a consequence of this result, we also have that groups factorized by totally permutable locally soluble subgroups are locally soluble. Moreover, it follows that any group totally factorized by two polycyclic subgroups is polycyclic (see [1, Theorem 4.4.2]).

Lemma 2.1. *Let the group $G = AB$ be the product of its totally permutable subgroups A and B . If A and B are soluble, then G is soluble.*

Let the group $G = AB$ be the product of two totally permutable subgroups A and B ; if X is a subgroup of A and Y is a subgroup of B , it is clear that the group

$H = XY$ is totally factorized by its subgroups X and Y . Thus, it follows easily that any product of two totally permutable locally supersoluble subgroups is likewise locally supersoluble, since it is well known that a polycyclic group is supersoluble provided that all its finite homomorphic images are supersoluble (see [3]).

Lemma 2.2. *Let the group $G = AB$ be the product of its mutually permutable subgroups A and B , and let W be a finite subset of G . Then there exists a finitely generated subgroup E of G such that $W \subseteq E$ and $E = (A \cap E)(B \cap E)$ is the product of its mutually permutable subgroups $A \cap E$ and $B \cap E$.*

Proof. Let U and V be finitely generated subgroups of A and B , respectively, such that W is contained in $E = \langle U, V \rangle$. As E is contained in $UB = BU$, we have $E = U(B \cap E)$ and so $E = (A \cap E)(B \cap E)$. Let X be any subgroup of $A \cap E$; then $XB = BX$, so the product $X(B \cap E) = XB \cap E$ is a subgroup of G and hence $X(B \cap E) = (B \cap E)X$. Similarly, we have $(A \cap E)Y = Y(A \cap E)$ for all subgroups Y of $B \cap E$. Therefore, the finitely generated subgroup E is factorized by its mutually permutable subgroups $A \cap E$ and $B \cap E$. \square

The above lemma has the following consequence, which generalizes the quoted theorem of Asaad and Shaalan [2] on products of mutually permutable subgroups.

Corollary 2.3. *Let the locally finite group $G = AB$ be the product of its mutually permutable subgroups A and B . If A is locally nilpotent and B is locally supersoluble, then G is locally supersoluble.*

For our purposes, we also need the following result of Beidleman and Heineken [5] concerning the behaviour of finite residuals of factors in a totally factorized group.

Lemma 2.4. *Let the periodic group $G = AB$ be the product of its totally permutable subgroups A and B . Then the finite residuals of A and B are normal subgroups of G .*

It is well known that if $G = AB$ is a finite soluble group factorized by its subgroups A and B , then for each set π of prime numbers, there exist Sylow π -subgroups X of A and Y of B such that the product XY is a Sylow π -subgroup of G (see [1, Lemma 1.3.2]).

Lemma 2.5. *Let the periodic group $G = AB$ be the product of its totally permutable subgroups A and B . If π is a set of prime numbers, and A_π and B_π are normal Sylow π -subgroups of A and B , respectively, then $A_\pi B_\pi$ is a Sylow π -subgroup of G .*

Proof. Assume for a contradiction that the π -subgroup $X = A_\pi B_\pi$ is properly contained in a π -subgroup P of G , and consider an element y of $P \setminus X$. Let U and V be finite subgroups of A and B , respectively, such that y belongs to UV , and put $U_\pi = U \cap A_\pi$ and $V_\pi = V \cap B_\pi$. Then $U_\pi V_\pi$ is a Sylow π -subgroup of the finite group UV ; on the other hand, $U_\pi V_\pi$ is properly contained in the π -subgroup $\langle U_\pi, V_\pi, y \rangle$ of UV . This contradiction proves the lemma. \square

We can now prove the main result of the paper.

Proof of Theorem B. Assume for a contradiction that the subgroup $L = \bigcap_{n \in \mathbb{N}} \gamma_n(G)'$ is not trivial. As the group G is locally supersoluble by Corollary 2.3, its commutator subgroup G' is locally nilpotent. Thus,

$$L = \text{Dr}_{p \in \mathbb{P}} L_p$$

is the direct product of its Sylow subgroups, and hence there exists a prime number p such that L_p is a non-trivial normal subgroup of G . Consider the largest normal p' -subgroup $O_{p'}(G)$ of G ; then $LO_{p'}(G)/O_{p'}(G)$ is a non-trivial subgroup of

$$\bigcap_{n \in \mathbb{N}} \gamma_n(G/O_{p'}(G))'$$

and so the factor group $G/O_{p'}(G)$ is likewise a counterexample. Therefore, it can be assumed without loss of generality that G contains no non-trivial normal p' -subgroups and in particular G' is a p -group. Moreover, the finite residuals $J(A)$ of A and $J(B)$ of B are normal subgroups of G by Lemma 2.4, and hence both $J(A)$ and $J(B)$ are p -groups. It follows that for each prime number $q \neq p$, the groups A and B have finite Sylow q -subgroups A_q and B_q , respectively.

Consider a prime number $q > p$. If y is any element of B_p , the finite group $A_q \langle y \rangle$ is supersoluble and so A_q is a normal subgroup of $A_q \langle y \rangle$. On the other hand, $A_q B_{p'}$ is a p' -group and hence it is abelian; therefore A_q is normal in $G = AB$, and so $A_q = \{1\}$. A similar argument shows that B_q is also trivial, so G has no elements of order q by Lemma 2.5. Thus, p is the largest prime in the set $\pi(G)$, and in particular, $\pi(G)$ is finite. It follows that both indices $|A : J(A)|$ and $|B : J(B)|$ are finite, so the subgroup $J = J(A)J(B)$ has finite index in G and it is of course the finite residual of G . Moreover, the factor group G/J cannot be nilpotent and so $p > 2$. Note also that J is abelian, since any group factorized by two Prüfer subgroups is obviously abelian.

Let P be a subgroup of type p^∞ of A and consider any element y of B_p . Then P is the finite residual of $P \langle y \rangle$, so P is normal in PB_p and hence $[P, B_p] = \{1\}$ because $p > 2$. On the other hand, P is contained in the centre of A_p , and hence $P \leq Z(A_p B_p)$. Similarly, every subgroup of type p^∞ of B lies in $Z(A_p B_p)$, so J is contained in $Z(A_p B_p)$ and the p -group $A_p B_p$ is nilpotent. As G' is not abelian, it follows from Itô's theorem that at least one of the factors A and B is not abelian. Suppose that A is not abelian, so A_p is also not abelian and hence it is covered by finite non-abelian subgroups. Let H be any finite non-abelian subgroup of A_p . If z is any element of $B_{p'}$ and H_0 is a subgroup of H , we have that H_0 is normal in the finite supersoluble group $H_0 \langle z \rangle$, so z induces a power automorphism on H and then $[H, z] = \{1\}$ (see [6, Hilfssatz 5]). It follows that $[A_p, B_{p'}] = \{1\}$. On the other hand, the group $A_{p'} B_{p'}$ is abelian and so $B_{p'}$ is a normal subgroup of $G = A_{p'} A_p B$. Thus, $B_{p'} = \{1\}$ and B is a p -group; in particular, the subgroup $A_p B$ is nilpotent. It follows that $A_{p'} \neq \{1\}$, so $A_{p'}$ is not normal in G and there exists an element b of B such that $[A_{p'}, b] \neq \{1\}$.

Let K be any finite subgroup of B . As the finite group $A_{p'}K$ is supersoluble, K is normal in $A_{p'}K$, and hence $A_{p'}$ acts as a non-trivial group of power automorphisms on B . It follows that the subgroup $\langle K, b \rangle$ must be abelian (see [6, Hilfsatz 5]), and so B itself is abelian. Let w be any element of B . As the finite group

$$E = \langle A_{p'}, b, w \rangle = A_{p'} \langle b, w \rangle$$

is not nilpotent, the nilpotent residual $\bar{\gamma}(E)$ of E is a non-trivial p -group and hence $\bar{\gamma}(E) = \langle b, w \rangle$ (see [4, Theorem 2]). Therefore, B is contained in $\gamma_\omega(G)$, so $\gamma_\omega(G) = (A \cap \gamma_\omega(G))B$ and $G = A\gamma_\omega(G)$. Moreover, the finite group G/J is the product of its nilpotent totally permutable subgroups AJ/J and BJ/J , so its nilpotent residual $\gamma_\omega(G)J/J$ is abelian (see [4, Theorem 1]) and hence $Z(G/J) \cap (\gamma_\omega(G)J/J) = \{1\}$ (see [9, 9.2.6 and 9.2.7]). It follows that

$$Z(AJ/J) \cap (\gamma_\omega(G)J/J) = \{1\},$$

and so

$$(AJ/J) \cap (\gamma_\omega(G)J/J) = \{1\}.$$

Thus, $A \cap \gamma_\omega(G)$ is a subgroup of J . On the other hand, $\gamma_\omega(G)$ is contained in the Sylow p -subgroup $A_p B$ of G and hence $A \cap \gamma_\omega(G) \leq Z(\gamma_\omega(G))$. Therefore,

$$\gamma_\omega(G) = (A \cap \gamma_\omega(G))B$$

is abelian and this last contradiction completes the proof of the theorem. \square

Proof of Theorem A. As G is locally finite, its residually nilpotent subgroups A and B are locally nilpotent. Moreover, for each prime number p , the Sylow p -subgroups of A and B satisfy the minimal condition, so they are nilpotent and hence also centrally \mathfrak{F} -perfect. Thus, the statement is a direct consequence of Theorem B. \square

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