BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 34(2), 219-229

Groups whose non-subnormal subgroups have a transitive normality relation

F. de Giovanni, A. Russo and G. Vincenzi

Abstract. This article investigates the structure of groups in which every subgroup either is subnormal or has a transitive normality relation, with special attention to the case in which subnormal subgroups have bounded defect.

Keywords: Subnormal subgroup; T-group

Mathematical subject classification: 20E15.

1 Introduction

The structure of groups for which the set of non-normal subgroups is small in some sense has been investigated by several authors. In particular, Romalis and Sesekin ([13],[14],[15]) considered soluble groups whose non-abelian subgroups are normal (*metahamiltonian groups*), proving that such groups have derived length at most 3 and finite commutator subgroup with prime power order.

A group G is said to be a *T-group* if all its subnormal subgroups are normal, i.e. if normality in G is a transitive relation. The structure of finite soluble T-groups has been described by Gaschütz [4], while Robinson [10] investigated infinite soluble groups with the property T. Recently, attention has been given to groups in which many subgroups have a transitive normality relation (see for instance [2]), and in particular the last two authors [17] considered groups in which all non-normal subgroups are T-groups.

The aim of this article is to study groups in which every non-subnormal subgroup has the property T. For each non-negative integer k, we shall denote by $\mathfrak{X}_k$ the class of all groups in which subgroups are either T-groups or subnormal with defect at most k. A famous example by Heineken and Mohamed (see [6]) proves that there exist metabelian periodic groups with all subgroups subnormal, but with no bounds for the defects of the subgroups. It follows easily that

Received 18 October 2001.

$\bigcup_k \mathfrak{X}_k$ is properly contained in the class $\mathfrak{X}_\infty$ consisting of groups in which all non-subnormal subgroups are T-groups. Note that $\mathfrak{X}_1$ is precisely the class of groups studied in [17].

In Section 2, among other solubility results, we prove that $\mathfrak{X}_\infty$-groups having an ascending series with locally (soluble-by-finite) factors are soluble, and that such a group has derived length at most 4, provided that it is not locally nilpotent. The last section is devoted to the structure of $\mathfrak{X}_k$-groups. In particular, it is shown that any periodic locally graded $\mathfrak{X}_k$-group is either finite-by-nilpotent or a T-group. Moreover, we prove that if G is a non-periodic locally soluble $\mathfrak{X}_k$-group, then each non-abelian subgroup of G is subnormal with defect at most k. Groups with this latter property are also studied, and it is proved that the results of Romalis and Sesekin on metahamiltonian groups can be extended to this case.

Most of our notation is standard and can be found in [12]. In particular, if H is a subgroup of a group G, the *series of normal closures* of H in G is defined inductively by the positions $H^{G,0} = G$ and $H^{G,k+1} = H^{H^{G,k}}$ for each non-negative integer k. Thus H is subnormal in G with defect k if and only if $H^{G,k} = H$ and k is the smallest non-negative integer with this property.

2 $\mathfrak{X}_\infty$-groups

Clearly the class $\mathfrak{X}_\infty$ (as well as the class $\mathfrak{X}_k$ for each non-negative integer k) is closed with respect to subgroups and homomorphic images. Moreover, as subnormal subgroups of T-groups are likewise T-groups, we have that every T-group in the class $\mathfrak{X}_\infty$ is a $\bar{T}$-group, i.e. a group in which all subgroups have the property T. Therefore the following lemma holds.

Lemma 2.1. *Let G be an $\mathfrak{X}_\infty$-group (respectively: an $\mathfrak{X}_k$-group for some $k \geq 0$). Then every subgroup of G is either subnormal (respectively: subnormal with defect at most k) or a $\bar{T}$-group.*

Lemma 2.2. *Let G be a finite $\mathfrak{X}_\infty$-group. Then G is soluble.*

Proof. As finite $\bar{T}$-groups are soluble, it can be assumed that the group G contains a proper non-trivial normal subgroup N. Then the groups N and G/N belong to the class $\mathfrak{X}_\infty$, and hence they are soluble by induction on the order of G. Therefore G is a soluble group. □

By a relevant theorem of Roseblade [16], there exists a function f such that if G is any group in which all subgroups are subnormal with defect bounded by a non-negative integer k, then G is nilpotent with class at most $f(k)$.

Lemma 2.3. *Let G be a group, and let H be a subgroup of G such that all subgroups of G containing H are subnormal with defect at most k. Then H contains the subgroup $G^{(n)}$, where $n = k([log_2 f(k)] + 1)$ and f is the function of Roseblade's theorem.*

Proof. For each non-negative integer $i < k$, all subgroups of the group $H^{G,i}/H^{G,i+1}$ are subnormal with defect at most k, so that $H^{G,i}/H^{G,i+1}$ is nilpotent of class at most $f(k)$ by Roseblade's theorem, and hence it has derived length at most $[log_2 f(k)]+1$. Since $H^{G,k} = H$, it follows that $G^{(n)}$ is contained in H. □

The example of Heineken and Mohamed quoted in the introduction shows that a group whose subgroups are subnormal need not be nilpotent. On the other hand, Möhres [7] proved that such groups must be soluble. Therefore the argument of the proof of Lemma 2.3 can also be used to obtain the following result.

Lemma 2.4. *Let G be a group, and let H be a subgroup of G such that all subgroups of G containing H are subnormal. Then there exists a non-negative integer n such that $G^{(n)}$ is contained in H.*

Corollary 2.5. *Let G be a perfect $\mathfrak{X}_\infty$-group. Then every proper subgroup of G is a T-group.*

Proof. Since $G = G^{(n)}$ for each non-negative integer n, it follows from Lemma 2.4 that all proper subgroups of G are T-groups. □

A group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Obviously, all locally (soluble-by-finite) groups are locally graded; it is also clear that every group having no infinite simple sections is locally graded.

Corollary 2.6. *Let G be a perfect locally graded $\mathfrak{X}_\infty$-group. Then G is a $\bar{T}$-group.*

Proof. By Corollary 2.5 every proper subgroup of G is a T-group. Assume by contradiction that G is not a $\bar{T}$-group. Since the property $\bar{T}$ is local, it follows that G is finitely generated, so that it has a finite non-trivial homomorphic image $\bar{G}$. The group $\bar{G}$ is soluble by Lemma 2.2, and this contradiction proves the corollary. □

Corollary 2.7. *Let G be an $\mathfrak{X}_\infty$-group having no infinite simple sections. Then G is hypoabelian and $G^{(\omega+2)} = \{1\}$.*

Proof. It is well known that $\bar{T}$-groups having no infinite simple sections are metabelian, so that it follows from Corollary 2.6 that G does not contain perfect non-trivial subgroups. Thus G is hypoabelian. Suppose that G is not soluble, so that also the factor group $G/G^{(\omega)}$ is not soluble. By Möhres' theorem there exists a non-subnormal subgroup of G containing $G^{(\omega)}$, so that $G^{(\omega)}$ is a $\bar{T}$-group and hence $G^{(\omega+2)} = \{1\}$. □

Since periodic locally graded $\bar{T}$-groups are metabelian, the argument used in the proof of Corollary 2.7 also shows that periodic locally graded $\mathfrak{X}_\infty$-groups are hypoabelian with length at most $\omega + 2$.

Recall that a subgroup H of a group G is said to be *pronormal* if for each element g of G the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. It is well known that groups rich of pronormal subgroups are closely related to T-groups.

The consideration of the alternating group of degree 4 shows that finite $\mathfrak{X}_\infty$-groups need not be supersoluble. On the other hand, as a consequence of our next result it turns out that such groups must have a Sylow tower.

Theorem 2.8. *Let G be a non-trivial locally finite $\mathfrak{X}_\infty$-group. Then G contains a non-trivial normal Sylow subgroup.*

Proof. Assume by contradiction that the theorem is false, and let p be any prime in the set $\pi(G)$. If P is a Sylow p-subgroup of G, there exist elements $x \in P$ and $y \in G$ such that x^y does not belong to P. Thus $\langle P, x^y \rangle$ is not a p-group, and so P contains a finite subgroup E such that $x \in E$ and $\langle E, x^y \rangle$ is not a p-group. Let a and g be elements of G such that the order of a is a power of p. Consider the finite soluble group $X = \langle E, y, a, g \rangle$, and let Y be a Sylow p-subgroup of X containing E. Clearly Y is not normal in X, and so the normalizer $N_X(Y)$ is not subnormal in X. Then $N_X(Y)$ must be a T-group, so that all subgroups of Y are normal in $N_X(Y)$, and hence $\langle a \rangle$ is a pronormal subgroup of X (see [12], p.298). In particular, the subgroups $\langle a \rangle$ and $\langle a^g \rangle$ are conjugate in $\langle a, a^g \rangle$, and so $\langle a \rangle$ is pronormal also in G. Therefore all cyclic subgroups of G are pronormal, and G is a $\bar{T}$-group (see for instance [5], Lemma 3.5). It follows that $[G', G]$ is an abelian Hall subgroup of G (see [12], p.405), and hence G contains a non-trivial normal Sylow subgroup. This contradiction proves the theorem. □

Our next lemma shows in particular that soluble $\mathfrak{X}_\infty$-groups are locally polycyclic.

Lemma 2.9. *Let G be a finitely generated soluble $\mathfrak{X}_\infty$-group. Then G is either nilpotent or abelian-by-finite.*

Proof. Suppose that the group G is not nilpotent, so that it must contain a non-subnormal subgroup A of finite index (see [12], p.477). Clearly A is a finitely generated soluble T-group, so that A is either finite or abelian, and hence G is abelian-by-finite. □

Theorem 2.10. *Let G be an $\mathfrak{X}_\infty$-group having an ascending series with locally (soluble-by-finite) factors. Then G is soluble. Moreover, if G is not locally nilpotent, then it has derived length at most* 4.

Proof. Let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending series of G whose factors are locally (soluble-by-finite). Assume that the group G is not soluble, and let $\mu \leq \tau$ be the least ordinal such that G_μ is not soluble, so that G_α is soluble for each ordinal $\alpha < \mu$. If μ is a limit ordinal, then

$$G_\mu = \bigcup_{\alpha<\mu} G_\alpha$$

is locally soluble; on the other hand, if μ is not a limit, the factor group $G_\mu/G_{\mu-1}$ is locally soluble by Lemma 2.2, and hence G_μ is locally soluble also in this case. Since soluble T-groups are metabelian, the group G_μ must contain a finitely generated subgroup E which is not a T-group. It follows from Lemma 2.1 that all subgroups of G containing E are subnormal, and so by Lemma 2.4 there is a positive integer n such that $G^{(n)}$ is contained in E. This contradiction proves that G is soluble. Suppose now that G is not locally nilpotent, and assume by contradiction that $G^{(4)} \neq \{1\}$, so that G contains a finitely generated non-nilpotent subgroup K with $K^{(4)} \neq \{1\}$. Since K is polycyclic by Lemma 2.9, there exists a normal subgroup L of K such that K/L is a finite non-nilpotent group with derived length at least 5. Replacing G by K/L, it can be assumed without loss of generality that G is a finite non-nilpotent group, and of course G can be chosen of smallest possible order. It follows from Theorem 2.8 that G contains a non-trivial normal Sylow subgroup P, and $G = PQ$ where Q is a subgroup of G with $P \cap Q = \{1\}$. If Q is subnormal in G, then $G = P \times Q$, so that Q is not nilpotent and $Q^{(4)} = \{1\}$; in this case P'' is not a T-group and G/P'' must be nilpotent, a contradiction by P.Hall's nilpotency criterion. Thus the subgroup Q is not subnormal, so that it is a T-group and G'' is contained in P. Clearly G'' is not a T-group, so that G/G'' is nilpotent and Q is a Dedekind group. As G is not nilpotent, it follows that also its commutator subgroup G' is not nilpotent, so that in particular G' is not contained in P. Thus Q is a hamiltonian group and P has odd order. Again the nilpotency criterion of P. Hall yields that $G/G^{(3)}$ is not nilpotent, so that the subgroup $G^{(3)}$ of P is a T-group, and hence it is abelian. This last contradiction completes the proof of the theorem. □

The proof of Theorem 2.10 also shows that under the additional assumption that the group G has no sections isomorphic to the quaternion group of order 8, the derived length of G is at most 3.

3 $\mathfrak{X}_k$-groups

We begin this section with the following solubility criterion for periodic $\mathfrak{X}_k$-groups. This is a consequence of Theorem 2.10.

Corollary 3.1. *Let G be a periodic $\mathfrak{X}_k$-group having an ascending series with locally graded factors. Then G is soluble.*

Proof. By Theorem 2.10 it is enough to prove that every periodic locally graded $\mathfrak{X}_k$-group G is soluble. Put $n = k([log_2 f(k)] + 1)$, where f is the function of Roseblade's theorem. If H is any subgroup of G which is not a T-group, then all subgroups of G containing H are subnormal with defect at most k, and hence $G^{(n)}$ is contained in H by Lemma 2.3. It follows that every proper subgroup of $G^{(n)}$ is a T-group, so that either $G^{(n)}$ is a $\bar{T}$-group or it is finitely generated. In any case, $G^{(n)}$ contains a subgroup of finite index which is a $\bar{T}$-group, and hence $G^{(n)}$ is soluble by Lemma 2.2. Therefore G is a soluble group. □

Let G be an $\mathfrak{X}_k$-group, where k is any non-negative integer. If H is a subnormal subgroup of G and $H^{G,k+1} \neq H$, the subgroup $H^{G,k+1}$ is subnormal in G with defect $k+1$, so that it is a T-group, and hence H is normal in $H^{G,k+1}$. It follows that every subnormal subgroup of an $\mathfrak{X}_k$-group has defect at most $k+2$.

Lemma 3.2. *Let G be a locally nilpotent $\mathfrak{X}_k$-group. Then every subgroup of G is subnormal with defect at most $k+2$, and in particular G is nilpotent with class at most $f(k+2)$, where f is the function of Roseblade's theorem.*

Proof. Let X be any subgroup of G which is not subnormal with defect at most k. Then X is a $\bar{T}$-group, and so even a Dedekind group. Clearly every join of a chain of Dedekind subgroups of G is likewise a Dedekind subgroup, and so by Zorn's Lemma X is contained in a maximal Dedekind subgroup M of G. Let $\mathfrak{L}$ be the set of all subgroups of G properly containing M, and let H be any element of $\mathfrak{L}$. Then H is not a $\bar{T}$-group, and hence it is subnormal in G with defect at most k. It follows that also the intersection L of all members of $\mathfrak{L}$ is a subnormal subgroup of G with defect at most k. On the other hand, either $M = L$ or M is a maximal subgroup of L, so that in particular M is normal in L. Therefore X is subnormal in G with defect at most $k + 2$. □

We need the following extension of the theorem of Roseblade, that has been recently obtained by E. Detomi (see [3], Theorem 3).

Lemma 3.3. *Let G be a periodic group for which the set $\pi(G)$ is finite. If there exists a finite subgroup E of G such that all subgroups of G containing E are subnormal with defect at most k for some fixed non-negative integer k, then G is finite-by-nilpotent.*

Theorem 3.4. *Let G be a periodic locally graded $\mathfrak{X}_k$-group. Then either G is a $\bar{T}$-group or it is finite-by-nilpotent.*

Proof. It follows from Corollary 3.1 that the group G is soluble, and so also locally finite. Suppose that G is not a $\bar{T}$-group, so that it contains a finite subgroup E which is not a T-group. Let π be the set of prime divisors of the order of E. Since every subgroup of G containing E is subnormal with defect at most k, the normal closure E^G of E is a π-group and the factor group G/E^G is nilpotent by Roseblade's theorem. By Lemma 3.2 it can be assumed that G is not locally nilpotent, so that there exists a prime number p such that the Sylow p-subgroups of G are not normal. Let K/E^G be the unique Sylow p-subgroup of G/E^G. Then K is normal in G and the set $\pi(K)$ is finite, so that K is finite-by-nilpotent by Lemma 3.3 and hence it contains a finite normal subgroup L of G such that K/L is nilpotent. Let P be any Sylow p-subgroup of G; then P is contained in K and PL/L is the unique Sylow p-subgroup of K/L. In particular, PL is normal in G and G/PL is a p'-group, so that all Sylow p-subgroups of G are contained in PL. Since the Sylow p-subgroups of PL are pairwise conjugate, it follows from Frattini's argument that $G = LN_G(P)$. On the other hand, $N_G(P)$ is a proper selfnormalizing subgroup of G, and hence $N_G(P)$ is a T-group; thus G/L is a T-group, and so the finite subgroup EL is normal in G. Therefore E^G is contained in EL, so that E^G is finite and the group G is finite-by-nilpotent. The theorem is proved. □

Let k be a non-negative integer. A group G is called *k-metahamiltonian* if every non-abelian subgroup of G is subnormal with defect at most k. Obviously, any k-metahamiltonian group belongs to the class $\mathfrak{X}_k$, and 1-metahamiltonian groups are precisely the metahamiltonian groups considered by Romalis and Sesekin.

Let G be a group. Recall that the *FC-centre* of G is the characteristic subgroup consisting of all elements of G having finitely many conjugates, and that G is an *FC-group* if it coincides with its FC-centre.

Theorem 3.5. *Let G be a locally soluble non-periodic $\mathfrak{X}_k$-group. Then G is k-metahamiltonian.*

Proof. Let X be any non-abelian subgroup of G. If X is not periodic, then it is not a $\bar{T}$-group, and so X is subnormal in G with defect at most k. Suppose now that X is periodic, and let x be any element of the subgroup $X^{G,k}$. Then there exists a finite non-abelian subgroup Y of X such that x belongs to $Y^{G,k}$. Let a be an element of infinite order of G, and put $E = \langle Y, a\rangle$. It follows from Lemma 2.9 that E is polycyclic. Let u be any element of E such that the cyclic subgroup $\langle u\rangle$ is not subnormal in E. Since $\langle u\rangle$ is intersection of subgroups of finite index of E (see [18], p.18), there exists a subgroup of finite index U of E such that $u \in U$ and U is not subnormal with defect at most k. Thus U is a finitely generated T-group, so that U is abelian and hence it is contained in the centralizer $C_E(u)$. It follows that u has finitely many conjugates in E, and hence $E = F \cup C$, where F is the Fitting subgroup and C is the FC-centre of E. Therefore E either is nilpotent or an FC-group. As finitely generated FC-groups are central-by-finite, we obtain that the centre $Z(E)$ of E contains an element of infinite order b. Then Y is a characteristic subgroup of $\langle Y, b\rangle = Y \times \langle b\rangle$; moreover, $\langle Y, b\rangle$ is not a T-group, so that it is subnormal in G with defect at most k, and hence also Y is subnormal with defect at most k. Therefore $Y^{G,k} = Y \leq X$, so that $X = X^{G,k}$ is subnormal in G with defect at most k, and G is a k-metahamiltonian group. □

A finite group G is called a *Δ-group* if $Z(G) = \{1\}$ and each subgroup of G is either subnormal or nilpotent. It has been proved by Phillips and Wilson [9] that a finite non-trivial group G is a Δ-group if and only if it is a semidirect product $G = C \ltimes A$, where A is an abelian minimal normal subgroup of G and C is a cyclic non-trivial subgroup such that each non-trivial element of C acts irreducibly on A by conjugation. Therefore all Δ-groups are metabelian. It also follows that every finite k-metahamiltonian group is nilpotent-by-cyclic, and in particular it has nilpotent commutator subgroup.

Our last theorem extends the results of Romalis and Sesekin on metahamiltonian groups to k-metahamiltonian groups for any non-negative integer k.

Theorem 3.6. *Let G be a locally graded k-metahamiltonian group which is not nilpotent. Then G is soluble with derived length at most* 3, *the factor group*

$G/Z(G)$ is finite and the commutator subgroup G' of G is finite with prime-power order.

Proof. Let H be any non-abelian subgroup of G. Then every subgroup of G containing H is subnormal with defect at most k, and hence it follows from Lemma 2.3 that the subgroup $G^{(n)}$ is contained in H, where

$$n = k\big([log_2 f(k)] + 1\big).$$

Therefore every proper subgroup of $G^{(n)}$ is abelian, and so $G^{(n)}$ contains an abelian subgroup of finite index. As finite k-metahamiltonian groups are soluble, it follows that the group G is soluble. By Lemma 3.2 we have that G is not locally nilpotent, and in order to prove that the index $|G : Z(G)|$ is finite, it can be assumed that G is a Černikov group (see [9], Theorem C(i)). Let E be a finite non-abelian subgroup of G. Then every subgroup of G containing E is subnormal with defect at most k, and in particular the factor group G/E^G is nilpotent. On the other hand, E has finitely many conjugates in G (see [11] Part 1, Theorem 5.49), so that E^G is finite and G is finite-by-nilpotent. It follows that $G/Z(G)$ is finite, so that also G' is finite by Schur's theorem. Let X be a finitely generated subgroup of G such that $G = XZ(G)$. Clearly $Z(X) = X \cap Z(G)$ is finitely generated, and so it contains a torsion-free subgroup K such that X/K is finite. Since X is not nilpotent and $G' = X' \simeq X'K/K$, replacing G by X/K it can be assumed without loss of generality that the group G is finite. It follows from Theorem 2.8 that G contains a non-trivial normal Sylow p-subgroup P for some prime number p, and so there exists also a subgroup Q of G such that $G = PQ$ and $P \cap Q = \{1\}$. If Q is abelian, then $G' \leq P$ has order a power of p. Suppose now that Q is not abelian, so that Q is subnormal in G and $G = P \times Q$. Thus Q is not nilpotent, and so it contains a non-subnormal subgroup L. It follows that PL is not subnormal in G, so that P is abelian and by induction on the order of G we obtain that $G' = Q'$ has prime-power order. Finally, since G' is nilpotent, the factor group G/G'' cannot be nilpotent, so that G'' is abelian and G has derived length at most 3. □

Corollary 3.7. *Let G be a locally soluble $\mathfrak{X}_k$-group. Then either G is a $\bar{T}$-group or it is finite-by-nilpotent.*

It has been proved by S.N. Černikov (see [1], Theorem 2.1) that every soluble non-periodic metahamiltonian group is metabelian. This result cannot be extended to k-metahamiltonian groups for $k \geq 2$. In fact, let $G = \langle x \rangle \ltimes Q$ be the semidirect product of a normal subgroup Q isomorphic to the quaternion group of order 8 and an infinite cyclic subgroup $\langle x\rangle$ such that x acts on Q as an automorphism of order 3; if X is any non-abelian subgroup of G, then $|XQ : X| \leq 2$ and hence X is subnormal in G with defect at most 2. Therefore G is a soluble non-periodic 2-metahamiltonian group with derived length 3.

Acknowledgments. We would like to thank professor Yaroslav P. Sysak for his useful comments and suggestions.

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F. de Giovanni
Dipartimento di Matematica e Applicazioni
Università di Napoli
via Cintia, I - 80126 Napoli
ITALY
E-mail: degiova@matna2.dma.unina.it

A. Russo
Dipartimento di Matematica
Università di Lecce
via Provinciale Lecce - Arnesano
P.O. Box 193, I - 73100 Lecce
ITALY
E-mail: alessio.russo@unile.it

G. Vincenzi
Dipartimento di Matematica e Informatica
Università di Salerno
via S. Allende, I - 84081 Baronissi, Salerno
ITALY
E-mail: gvincenzi@unisa.it