New solutions of the Yang-Baxter equation obtained through solutions of the pentagon equation

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The study of the pentagon equation (PE) classically originates from the field of Mathematical Physics and it is widely investigated also in Analysis. The paper [Dimakis, Müller-Hoissen, 2015] can be useful for a brief introduction to this topic.

Recent developments have been provided in [Catino, Mazzotta, Miccoli, 2019], where this equation is dealt with from an algebraic point of view.

**Aim of this talk**

Show new applications of the PE to set-theoretical solutions of the well-known Yang-Baxter equation. [Catino, Mazzotta, S., work in progress]
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Given a set $S$, a map $s : S \times S \to S \times S$ is a set-theoretical solution of the PE on $S$ if

$$s_{23} s_{13} s_{12} = s_{12} s_{23}$$

where $s_{12} = s \times \text{id}_S$, $s_{23} = \text{id}_S \times s$, and $s_{13} = (\text{id}_S \times \tau)s_{12}(\text{id}_S \times \tau)$ with $\tau$ the twist map, i.e., $\tau(x, y) = (y, x)$, for all $x, y \in S$.

We briefly call $s$ a solution of the PE.

In particular, as in [Catino, Mazzotta, Miccoli, 2019] we write

$$s(a, b) = (a \cdot b, \theta_a(b)),$$

for all $a, b \in S$, where $\theta_a$ is a map from $S$ into itself, for every $a \in S$. Note that the structure $(S, \cdot)$ is a semigroup.
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Given a set $S$, a map $s : S \times S \rightarrow S \times S$ is a *set-theoretical solution of the PE* on $S$ if

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A comparison with the PE:

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**Remark:** A map $s$ is a solution of the PE if and only if $t := \tau s \tau$ is a reversed solution, that is given by

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Examples

- If $S$ is a semigroup and $\gamma$ an idempotent endomorphism of $S$ then the map $s : S \times S \rightarrow S \times S$ given by

$$s(a, b) = (ab, \gamma(b))$$

is a solution of the PE on $S$ but not of the R-PE.

- Militaru solutions: Given $f$ and $g$ idempotent maps from a set $S$ into itself such that $fg = gf$. Then the map $s$ given by

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Set-theoretical solutions of the quantum YBE

According to [Drinfel’d, 1992], given a set \( S \), a map \( \mathcal{R} : S \times S \rightarrow S \times S \) is said to be a set-theoretical solution of the quantum Yang-Baxter equation on \( S \), if

\[
\mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} = \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23}
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holds, with the same notation adopted for the PE. For simplicity, we call \( \mathcal{R} \) a solution of the QYBE.

Comparison with the QYBE:

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The pentagon equation and the Yang-Baxter equation

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A special class of solutions

Proposition (Catino, Mazzotta, S., 2019)

Let \( s \) be a solution of the PE on a set \( S \) defined by \( s(a, b) = (ab, \theta_a(b)) \). Then, the map \( s \) is a solution of the QYBE if and only if the following conditions

\[
abc = a\theta_b(c)bc \quad (Y1)
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are satisfied, for all \( a, b, c \in S \). We call \( s \) a solution to the QYBE of pentagonal type, or briefly a solution P-QYBE.

Analogously, if \( t \) is a reversed solution, \( t(a, b) = (\theta_b(a), ba) \), then \( t \) is a solution of the QYBE if and only if \((Y1), (Y2), \) and \((Y3)\) are satisfied. We call \( t \) a solution to the QYBE of reversed pentagonal type, or briefly a solution R-QYBE.
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Analogously, if \( t \) is a reversed solution, \( t(a, b) = (\theta_b(a), ba) \), then \( t \) is a solution of the QYBE if and only if (Y1), (Y2), and (Y3) are satisfied. We call \( t \) a solution to the QYBE of reversed pentagonal type, or briefly a solution R-QYBE.
A special class of solutions

Proposition (Catino, Mazzotta, S., 2019)

Let $s$ be a solution of the PE on a set $S$ defined by $s(a, b) = (ab, \theta_a(b))$. Then, the map $s$ is a solution of the QYBE if and only if the following conditions

1. $abc = a\theta_b(c)bc$ \hspace{1cm} (Y1)
2. $\theta_a\theta_b = \theta_b$ \hspace{1cm} (Y2)
3. $\theta_a(bc) = \theta_{\theta_b(c)}(bc)$ \hspace{1cm} (Y3)

are satisfied, for all $a, b, c \in S$. We call $s$ a solution to the QYBE of pentagonal type, or briefly a solution P-QYBE.

Analogously, if $t$ is a reversed solution, $t(a, b) = (\theta_b(a), ba)$, then $t$ is a solution of the QYBE if and only if (Y1), (Y2), and (Y3) are satisfied. We call $t$ a solution to the QYBE of reversed pentagonal type, or briefly a solution R-QYBE.
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Examples

- **Militaru solutions**: If \( f \) and \( g \) idempotent maps from a set \( S \) into itself such that \( fg = gf \), the map

\[
s(a, b) = (f(a), g(b))
\]

is a solution P-QYBE on \( S \). In this case the semigroup operation is defined by \( ab := f(a) \). Clearly, \( s \) lies in the class of the well-known Lyubashenko solutions.

- If \( S \) is such that \( abc = adbc \), for all \( a, b, c, d \in S \) (cf. [Monzo, 2003]), then

\[
s(a, b) = (ab, \gamma(b))
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with \( \gamma \) an idempotent endomorphism, is a solution to the P-QYBE on \( S \).

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is a solution of the PE on \( S \) but not of the YBE since (Y1) does not hold.
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Solutions of the P-YBE on particular semigroups

We focus on solutions \( s(a, b) = (ab, \theta_a(b)) \) of the PE defined on specific varieties of semigroups \( S \) with the property

\[
abc = adbc
\]

for all \( a, b, c, d \in S \). We are interested in analysing the powers of the solutions of the “braid version” of the P-QYBE, i.e.,

\[
r(a, b) := \tau s(a, b) = (\theta_a(b), ab).
\]

Given a set \( S \), a map \( \mathcal{R} : S \times S \to S \times S \) into itself is a solution to the QYBE on \( S \) if and only if the map \( r := \tau \mathcal{R} \) satisfies the braid equation, i.e.,

\[
(r \times \text{id}_S)(\text{id}_S \times r)(r \times \text{id}_S) = (\text{id}_S \times r)(r \times \text{id}_S)(\text{id}_S \times r).
\]
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Given a set $S$, a map $R : S \times S \to S \times S$ into itself is a solution to the QYBE on $S$ if and only if the map $r := \tau R$ satisfies the braid equation, i.e.,

$$(r \times \text{id}_S)(\text{id}_S \times r)(r \times \text{id}_S) = (\text{id}_S \times r)(r \times \text{id}_S)(\text{id}_S \times r).$$
The powers of these solutions P-YBE

We show that the solutions P-YBE on these semigroups lie in a special class of solutions of the Yang-Baxter equation.

**Theorem (Catino, Mazzotta, S., 2019)**

Let $S$ be a semigroup with the property $abc = adbc$ and $r$ a (braid) solution P-YBE on $S$. Then, it holds

$$ r^5 = r^3 $$

and the powers $r^2$, $r^3$, $r^4$ of the map $r$ are still solutions to the YBE.

**Remark - Example**

If $S$ is a left quasi normal semigroup, i.e., $abc = acbc$, then the map on $S$ defined by $r(a, b) := (b, ab)$ is a solution of the P-YBE such that $r^5 = r^3$. If $S$ is not idempotent, then $r^2$, $r^3$, $r^4$ are not solutions of the YBE.

**Remark:** If $S$ is also idempotent, it holds $r^4 = r^2$. 
Solutions of the QYBE of pentagonal type

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Examples

- **Militaru solutions:** If $f$ and $g$ idempotent maps from a set $S$ into itself such that $fg = gf$, then the solution P-YBE defined by
  \[ r(a, b) = (g(b), f(a)) \]
is such that $r^4 = r^2$. Note that here $ab = f(a)$.

- If $S$ is such that $abc = adbc$, for all $a, b, c \in S$, then the solution to the P-YBE defined by
  \[ r(a, b) = (\gamma(b), ab) \]
with $\gamma$ idempotent endomorphism of $S$, is such that $r^5 = r^3$. 

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Examples

- **Militaru solutions**: If \( f \) and \( g \) idempotent maps from a set \( S \) into itself such that \( fg = gf \), then the solution P-YBE defined by

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  with $\gamma$ idempotent endomorphism of $S$, is such that $r^5 = r^3$. 
A new method to construct solutions to the YBE

We introduce a new method to construct solutions of the Yang-Baxter equation defined on the Cartesian product of two sets $S$ and $T$ through solutions of the pentagon equation.

In particular, we show how to obtain a solution of the YBE involving a solution $s$ of the PE and a solution $t$ of the R-YBE.
We introduce a new method to construct solutions of the Yang-Baxter equation defined on the Cartesian product of two sets $S$ and $T$ through solutions of the pentagon equation.

In particular, we show how to obtain a solution of the YBE involving a solution $s$ of the PE and a solution $t$ of the R-YBE.
A new construction - I

We introduce the following definition.

Definition

Let $S, T$ be semigroups, $s$ a solution of the PE on $S$ and $t$ a solution R-YBE on $T$. Let $\alpha : T \to S^S$ be a map, set $\alpha_u := \alpha(u)$, for every $u \in T$, and set

$$a_u b_v := \alpha_u(a) \alpha_{\theta_v(u)}(b),$$

for all $a, b \in S$ and $u, v \in T$. If the following conditions hold

$$a b_u c_v = a \theta_b \alpha_v(c) b_u c_v$$
$$\theta_a \theta_b \alpha_u = \theta_{\alpha_v(b)} \alpha_{\theta_u(v)}$$
$$\theta_a(b c) = \theta_{a \theta_b \alpha_u(c)}(bc)$$
$$a_u b_v = \alpha_{\theta_wv(u)}(a \alpha_v(b))$$
$$\theta_a = \alpha_u \theta_a$$

for all $a, b, c \in S$ and $u, v, w \in T$, then we call $(s, t, \alpha)$ a pentagon triple.
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$$\theta_a \theta_b \alpha_u = \theta_{\alpha_v(b)} \alpha_{\theta_u(v)}$$

$$\theta_a (bc) = \theta_{a \theta_b \alpha_u(c)}(bc)$$

$$a_u b_v = \alpha_{\theta_{uv}(u)}(a \alpha_v(b))$$

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$$\theta_a(bc) = \theta_a \theta_b \alpha_u(c)(bc),$$

$$a_u b_v = \alpha_{\theta_{wv}(u)}(a \alpha_v(b)),$$

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Let $S$, $T$ be semigroups, $s$ a solution of the PE on $S$ and $t$ a solution R-YBE on $T$. Let $\alpha : T \rightarrow S^S$ be a map, set $\alpha_u := \alpha(u)$, for every $u \in T$, and set

$$a_u b_v := \alpha_u(a) \alpha_{\theta_v(u)}(b),$$

for all $a, b \in S$ and $u, v \in T$. If the following conditions hold

$$a\ b_u\ c_v\ =\ a\theta_b\alpha_v(c)\ b_u\ c_v$$
$$\theta_a\theta_b\alpha_u\ =\ \theta_{\alpha_v(b)}\alpha_{\theta_u(v)}$$
$$\theta_a(bc)\ =\ \theta_{a\theta_b\alpha_u(c)}(bc)$$
$$a_u b_v\ =\ \alpha_{\theta_{w\nu}(u)}(a\alpha_v(b))$$
$$\theta_a\ =\ \alpha_u\theta_a$$

for all $a, b, c \in S$ and $u, v, w \in T$, then we call $(s, t, \alpha)$ a **pentagon triple**.
A new construction - I

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\]

\[
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\[
\theta_a(b c) = \theta_{a\theta_b \alpha_u(c)}(b c)
\]

\[
a_u b_v = \alpha_{\theta^w(v)}(a \alpha_v(b))
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for all $a, b, c \in S$ and $u, v, w \in T$, then we call $(s, t, \alpha)$ a **pentagon triple**.
A new construction - II

Theorem (Catino, Mazzotta, S., 2019)

Let \((s, t, \alpha)\) be a pentagon triple. Then the map given by

\[
r(a, u; b, v) = (\theta_a \alpha_u(b), vu; a\alpha_u(b), \theta_v(u)),
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for all \((a, u), (b, v) \in S \times T\) is a solution of the YBE.

This result is a special case of a more general construction.
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Example

Consider

- $S$ a semigroup with the properties $abdbc = abc$ and $a^3 = a^2$, $k \in S$, and $s(a, b) = (ab, k^2)$ the solution of the PE on $S$ (it is not a solution to the QYBE);

- $T$ a semigroup with the property $adbc = abc$ and $t(u, v) = (u, vu)$ a solution R-QYBE on $T$;

- $\alpha_u(a) = k^2$, for every $a \in S$ and $u \in T$.

Hence, the map given by

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Hence, the map given by

$$r(a, u ; b, v) = (k^2, vu ; ak^2, u),$$

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P. Stefanelli | New solutions of the YBE obtained through solutions of the PE
Thanks for your attention!