Interaction Between Convergence Spaces and Discrete Groups

Pranav Sharma
Lovely Professional University. India.

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**Edge-colored Cayley digraph**

**Generating set Γ of a group G**

Let Γ be a subset of a group G such that each element of G is a product of elements of Γ and no element of Γ is redundant.

**Edge-colored Cayley digraph**

The Cayley digraph for G generated by Γ is the directed graph C such that the vertex set of C is G and the edge set of C is $E = \{(g, g\gamma) : g \in G, \gamma \in \Gamma\}$. The edges are colored by $j : E \to \Gamma$, where $j(g, h) = s$.

Consider the symmetric group $S_3$ with generators $\{(12), (123)\}$.

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Cayley Graphs

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Edge-colored Cayley digraph of $S_3$
Cayley digraph (without color)
Reflexive Cayley digraph

The Cayley graph for $G$ generated by $\Gamma$ is the reflexive digraph $C$ such that the vertex set of $C$ is $G$ and the edge set of $C$ is $\{(g, h) : g\gamma = h \land (\gamma = e \lor \gamma \in \Gamma)\}$.

\[\text{id}\] (123)
(12) (13)
(132) (23)

\[\text{id}\] (123)
Convergence generated by a reflexive digraph

The graph neighbourhood\(^2\) of the vertices are
\[ 1 \rightarrow = \{1\}, \quad 2 \rightarrow = \{1, 2\}, \quad 3 \rightarrow = \{1, 2, 3\}. \]

This graph can be represented as the following convergence:

\[
\begin{align*}
\{1\}^\uparrow & \rightarrow \{1, 2, 3\} & \{1, 2\}^\uparrow & \rightarrow \{2, 3\} & \{1, 2, 3\}^\uparrow & \rightarrow \{3\} \\
\{2\}^\uparrow & \rightarrow \{2, 3\} & \{1, 3\}^\uparrow & \rightarrow \{3\} \\
\{3\}^\uparrow & \rightarrow \{3\} & \{2, 3\}^\uparrow & \rightarrow \{3\} 
\end{align*}
\]

No topology can describe this convergence.

Convergence space

Let $\lambda$ be an arbitrary relation between $X$ and the power set of, set of all filters on $X$. The relation is called convergence on that set if for $F_1, F_2$ in $\mathcal{F}X$ and $x$ in $X$ the following conditions hold:

(i) **Centred:** $x^\uparrow \in \lambda(x)$,
(ii) **Isotone:** If $F_1 \in \lambda(x)$ and $F_1 \leq F_2$ then $F_2 \in \lambda(x)$, and
(iii) **Finitely deep:** If $F_1, F_2 \in \lambda(x)$ then, $F_1 \cap F_2 \in \lambda(x)$.

Category of convergence spaces contain category of reflexive directed graphs.

Reflexive Cayley digraph of $\mathbb{Z}$ generated by $\{1\}$

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A Cayley graph for $\mathbb{Z} \oplus \mathbb{Z}$ is the graph Cartesian product $\mathbb{Z} \times \mathbb{Z}$ generated by $(\Gamma \times \{e\}) \cup (\{e\} \times \Gamma)$.

Define a function $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $+(a, b) = a + b$. Clearly, $+$ is continuous.

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[Example 4.6], Patten et. al, Differential calculus on Cayley graphs, 2015
**Convergence Groups**

A convergence group is a group endowed with a convergence structure such that group operations are continuous in the sense of convergence.

**Remark**

- The equivalence between the Cayley graphs and the convergence spaces can be used to construct the convergence groups beyond the class of homeomorphism groups.

- Convergence spaces are used in unifying discrete and continuous models of computation and they play a vital role in extending the definition of differential to the discrete structures.

- Convergence spaces play a prominent role in extending the Pontryagin duality theorem beyond local compactness.
Dual group

Circle group, \( \mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong U(1) \)

The multiplicative group of all complex numbers of unit modulus with the natural topology as a subspace of the complex plane.

Character group, \( \hat{G} \) or \( \text{CHom}(G, \mathbb{T}) \)

The homomorphism, \( \chi : G \rightarrow \mathbb{T} \) is called a character and the set of all continuous characters, of an abelian group with the operation of pointwise multiplication is called character group.

Dual group, \( (\hat{G}, \tau_{co}) \)

The character group with compact open topology is called the dual group.
Pontryagin-van Kampen theorem

Pontryagin duality

For a topological abelian group there is a natural evaluation homomorphism from the group to its double dual defined by

\[ \alpha_G : G \to \hat{\hat{G}} \quad \alpha_G(g)(\chi) = \chi(g) \quad \forall \; g \in G. \]

If this evaluation map is a topological isomorphism then the group is said to satisfy Pontryagin duality or is said to be **Pontryagin reflexive**.

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**Pontryagin-van Kampen theorem**

Every locally compact abelian (LCA) group is canonically isomorphic to its double dual group.

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Moving Beyond Topology?

A topology on $\mathcal{C}Hom(G, \mathbb{T})$ is called admissible if the evaluation mapping

$$e : \mathcal{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}, \quad e(\chi, g) = \chi(x)$$

is continuous.

Reflexive Admissible Topological Group

If $G$ is a reflexive topological abelian group, then the evaluation mapping is continuous if and only if $G$ is locally compact.

Topological structures are inadequate for situations in analysis particularly when we go beyond local compactness.

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Continuous convergence structure $\lambda_c$

The continuous convergence structure on the character group of a convergence abelian group is the coarsest convergence structure which makes the evaluation mapping $e : \mathcal{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}$ continuous.

Continuous dual group, $\Gamma_c G$

The character group with continuous convergence structure is called the dual group.

Continuous duality

A convergence group is $c$-reflexive if the mapping

$$\kappa : G \to \Gamma_c \Gamma_c G$$

defined by

$$\kappa(g)(\chi) = \chi(g) \quad \forall \, g \in G, \, \chi \in \Gamma_c G$$

is a continuous group homomorphism, here $\Gamma_c G = (\mathcal{C}Hom(G, \mathbb{T}), \lambda_c)$

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Example

**Non c-reflexive locally compact convergence group**

Let $X$ be a locally compact topological space, $C(X)$ and $C(X, \mathbb{T})$ respectively denote the group of all continuous, real-valued functions on $X$ and the group of unimodular ($X \to \mathbb{T}$) continuous functions on $X$. Define

$$\rho : C_c(X) \to C_c(X, \mathbb{T}) \text{ as } \rho(f) = \rho \circ f.$$ 

- $\rho$ is continuous and a group homomorphism.

As $X$ is locally compact so

$$C_c(X) = C_{co}(X) \text{ and } C_c(X, \mathbb{T}) = C_{co}(X, \mathbb{T}).$$

- $C_c(X, \mathbb{T})$ is reflexive.

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For
\[ \kappa : X \to \Gamma_c C(X, \mathbb{T}) \] defined as \( \kappa(x)f = f(x) \)
we have,

**Theorem** \(^9\)

The group generated by \( \kappa(X) \), (denoted \( G = < \kappa(X) > \)) is a locally compact subgroup of \( \Gamma_c C_c(X, \mathbb{T}) \).

**Example** \(^10\)

For \( X \) a connected, compact topological space, the group \( G = < \kappa(X) > \) is not reflexive.

**Problem**

- To characterise the class of **reflexive locally compact convergence groups**.

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Approach to solve the problem

- Local quasi convexity.
  - It has been obtained that local quasi convexity is a necessary condition for a locally compact convergence group to be c-reflexive.
  - Non-topological compact convergence groups (if they exist) are reflexive iff they are locally quasi convex.

- Convergence measure spaces.
  - The problem is to obtain the notation of integration on convergence spaces and hence a notation similar to Haar measure for the convergence groups.
  - Are the convergence groups whose topological modification locally compact topological group reflexive?

- Bounded convergence groups.