On contractions and degenerations of algebras

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Let $n \geq 3$. In this joint work with **N.M. Ivanova** we classify all n-dimensional algebras over an arbitrary infinite field which have the property that the n-dimensional Abelian Lie algebra is their only proper degeneration.

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1 Introduction

The concept of degeneration probably first arises in the second half of the twentieth century when a lot of attention was paid to the study of various limit processes linking physical theories. The pioneering works in this direction were a paper by Segal (1951) who studied a limit process of a family of some physically important isomorphic Lie groups, and a series of papers of Inönü and Wigner (1953, 1954) devoted to the limit process $c \to \infty$ in special relativity theory showing how the symmetry group of relativistic mechanics (the Poincaré group) degenerates to the symmetry group of classical mechanics (Galilean group). The target algebras of such limit processes (which are nothing else but the points in the closure in metric topology of the orbit of the initial algebra under the 'change of basis' action of the general linear group) are called contractions (or degenerations in the more general context of an arbitrary field and Zariski closure).

Despite their theoretical and practical interest, results about degenerations, especially in fields other then $\mathbb{C}$ or $\mathbb{R}$, are still fragmentary. In work of Grunewald and O'Halloran, it is shown that the closures in the Zariski topology and in the standard topology of the orbit of a point of an affine variety over $\mathbb{C}$ under the action of an algebraic group coincide. A criterion for a Lie algebra to be a degeneration of another Lie algebra over an algebraically closed field is given in other work of Grunewald and O'Halloran. In a paper of V.L. Popov the question of whether or not a given orbit (of a point under the action of an algebraic group) lies in the Zariski closure of another orbit is being considered and a method of solving this problem is presented. However, in practice it is extremely difficult to apply the above criterion and method.

For some classes of algebras (like real and complex 3- and 4-dimensional Lie algebras, low-dimensional nilpotent Lie algebras, some subclasses of Malcev algebras) the problem of determining all degenerations within the given class has been considered by various authors.
Our motivation comes from works of Gorbatsevich and in particular the notion of the level of complexity of a finite dimensional algebra.

It is a well-known fact that if $\mathbb{F}$ is infinite then every n-dimensional $\mathbb{F}$-algebra degenerates to $\mathfrak{a}_n$, the n-dimensional Abelian Lie algebra over $\mathbb{F}$. We will be concerned with the following

Problem. Determine the isomorphism classes of n-dimensional $\mathbb{F}$-algebras, where $\mathbb{F}$ is an arbitrary infinite field, that have $\mathfrak{a}_n$ as their only proper degeneration.

Note that the case $\mathbb{F} = \mathbb{C}$ is already settled by works of Gorbatsevich and Khudoyberdiyev and Omirov. There is also relevant work of Lauret over $\mathbb{F} = \mathbb{R}$.

2 Preliminaries

Fix a positive integer n with $n \geq 3$ and an arbitrary infinite field $\mathbb{F}$. Also let $G = \mathrm{GL}(n, \mathbb{F})$.

Definition. An n-dimensional $\mathbb{F}$-algebra (not necessarily associative) is a pair $(A, [,])$ where A is a vector space over $\mathbb{F}$ with $\dim_{\mathbb{F}} A = n$ and $[,] : A \times A \to A : (x, y) \mapsto [x, y]$ $(x, y \in A)$ is an $\mathbb{F}$-bilinear map. We call $[x, y]$ the product of x and y.

The *structure constants* of this algebra with respect to its ordered basis $(b_1, \ldots, b_n)$ are the scalars $\alpha_{ijk} \in \mathbb{F}$ $(1 \leq i, j, k \leq n)$ given by $[b_i, b_j] = \sum_{k=1}^{n} \alpha_{ijk} b_k$. We will regard this set of structure constants α_{ijk} as an ordered n^3-tuple $(\alpha_{ijk})_{1 \leq i,j,k \leq n}$ in $\mathbb{F}^{n^3}$ via the lexicographic ordering on the triples (i, j, k) for $1 \leq i, j, k \leq n$. We call the n^3-tuple $\boldsymbol{\alpha} = (\alpha_{ijk}) \in \mathbb{F}^{n^3}$ the structure vector of $(A, [,])$ relative to the $\mathbb{F}$-basis $(b_1, \ldots, b_n)$ of A. More generally, we call the element $\boldsymbol{\beta} = (\beta_{ijk})_{1 \leq i,j,k \leq n} \in \mathbb{F}^{n^3}$ a *structure vector* for $(A, [,])$ if there exists an ordered $\mathbb{F}$-basis $(b'_1, \ldots, b'_n)$ for A relative to which the structure vector for $(A, [,])$ is $\boldsymbol{\beta} = (\beta_{ijk})$.

For the rest of this talk we fix V to be an n-dimensional $\mathbb{F}$-vector space. We also fix $(v_1^*, \ldots, v_n^*)$ to be an ordered $\mathbb{F}$-basis of V. We will be referring to this basis of V on several occasions in the sequel.

Definition. We call $\mathfrak{g}$ an algebra structure on V if $\mathfrak{g}$ is an $\mathbb{F}$-algebra having V as its underlying vector space (and hence has multiplication defined via a suitable $\mathbb{F}$-bilinear map $[,]_{\mathfrak{g}} : V \times V \to V$). We denote by $\mathcal{A}_n(\mathbb{F})$ the set of all algebra structures on V.

Regarding $\mathcal{A}_n(\mathbb{F})$ and $\mathbb{F}^{n^3}$ as $\mathbb{F}$-vector spaces in the usual way, we can obtain an $\mathbb{F}$-isomorphism $\Theta : \mathcal{A}_n(\mathbb{F}) \to \mathbb{F}^{n^3}$ such that the image of an algebra structure $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ is the structure vector of $\mathfrak{g}$ relative to the basis $(v_1^*, \ldots, v_n^*)$ of V we have fixed.

Example. $\Theta(\mathfrak{a}_n) = \mathbf{0}$, the zero vector of $\mathbb{F}^{n^3}$.

Definition. With the help of the bijection $\Theta : \mathcal{A}_n(\mathbb{F}) \to \mathbb{F}^{n^3}$, given above we can define a map $\Omega : \mathbb{F}^{n^3} \times G \to \mathbb{F}^{n^3} : (\boldsymbol{\lambda}, g) \mapsto \boldsymbol{\lambda} g$ ($\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$, $g = (g_{ij}) \in G$) where $\boldsymbol{\lambda} g \in \mathbb{F}^{n^3}$ is the structure vector of $\Theta^{-1}(\boldsymbol{\lambda}) \in \mathcal{A}_n(\mathbb{F})$ relative to the basis $(v_1, \ldots, v_n)$ of V given by $v_j = \sum_{i=1}^{n} g_{ij} v_i^*$.

The map Ω defines a linear right action of G on $\mathbb{F}^{n^3}$. We can thus regard $\mathbb{F}^{n^3}$ as a right $\mathbb{F}G$-module via this action. The corresponding orbit for $\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$ will be denoted by $O(\boldsymbol{\lambda})$. Note that the resulting orbits correspond precisely to isomorphism classes of n-dimensional $\mathbb{F}$-algebras.

We will be concerned with algebraic (Zariski-closed) subsets of $\mathbb{F}^{n^3}$. The Zariski closure of a subset Y of $\mathbb{F}^{n^3}$ will be denoted by $\overline{Y}$.

Next we introduce some subsets of $\mathcal{A}_n(\mathbb{F})$ which are defined via identities.

(i) $\mathcal{B}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) : [[x_1, x_2], x_3] = 0_V \text{ for all } x_1, x_2, x_3 \in V\}$.

(ii) $\mathcal{K}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) : [x, x] = 0_V \text{ for all } x \in V\}$.

(iii) $\mathcal{L}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{K}_n^a(\mathbb{F}) : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0_V$ for all $x, y, z \in V\}$.

Denote by $\mathcal{B}_n(\mathbb{F})$, $\mathcal{K}_n(\mathbb{F})$, $\mathcal{L}_n(\mathbb{F})$ the subsets of $\mathbb{F}^{n^3}$ which are the images of $\mathcal{B}_n^a(\mathbb{F})$, $\mathcal{K}_n^a(\mathbb{F})$, $\mathcal{L}_n^a(\mathbb{F})$ respectively via the map Θ.

One can observe that $\mathcal{B}_n(\mathbb{F})$, $\mathcal{K}_n(\mathbb{F})$ and $\mathcal{L}_n(\mathbb{F})$ are all unions of orbits. Moreover, they are algebraic subsets of $\mathbb{F}^{n^3}$ as they can be described via polynomial equations.

Clearly $\mathcal{K}_n(\mathbb{F})$ is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$ ($\dim_{\mathbb{F}} \mathcal{K}_n(\mathbb{F}) = \frac{1}{2}n^2(n-1)$). As $\mathcal{K}_n(\mathbb{F})$ also is union of orbits, it can be regarded as an $\mathbb{F}G$-module via the (linear) action of G on $\mathbb{F}^{n^3}$ we are considering.

For the rest of this talk, by an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$ we will mean any subspace of $\mathbb{F}^{n^3}$ which is also an $\mathbb{F}G$-module via the above action of G.

3 Degenerations

Lemma. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{F}^{n^3}$ with $\mu \in \overline{O(\boldsymbol{\lambda})}$. Then $O(\boldsymbol{\mu}) \subseteq \overline{O(\boldsymbol{\lambda})}$ (and hence $\overline{O(\boldsymbol{\mu})} \subseteq \overline{O(\boldsymbol{\lambda})}$).

Much more can be said using the theory of algebraic groups if the field $\mathbb{F}$ is algebraically closed. In particular, we then have that the orbits are locally closed.

Definition. Let $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$. We say that $\mathfrak{g}_1$ degenerates to $\mathfrak{g}_2$ (respectively, $\mathfrak{g}_1$ properly degenerates to $\mathfrak{g}_2$) if there exist structure vectors $\boldsymbol{\lambda}_1$ of $\mathfrak{g}_1$ and $\boldsymbol{\lambda}_2$ of $\mathfrak{g}_2$ such that $\boldsymbol{\lambda}_2 \in \overline{O(\boldsymbol{\lambda}_1)}$ (respectively, $\boldsymbol{\lambda}_2 \in \overline{O(\boldsymbol{\lambda}_1)} \setminus O(\boldsymbol{\lambda}_1)$).

Our next lemma will play some part in the proof of our main result. The following definition is needed in order to state this lemma.

Definition. Let $\hat{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$ be given. Also let $T = \{(i,j,k) \in \mathbb{Z}^3 : 1 \leq i,j,k \leq n\}$.

(i) For each $r \in \mathbb{Z}$, define the subset $S(\hat{q}, r)$ of T by $S(\hat{q}, r) = \{(i,j,k) \in T : q_i + q_j - q_k = r\}$.

(ii) Given a structure vector $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$, define a new structure vector $\boldsymbol{\lambda}(\hat{q}) = (\lambda_{ijk}(\hat{q})) \in \mathbb{F}^{n^3}$ by

$$\lambda_{ijk}(\hat{q}) = \begin{cases} \lambda_{ijk}, & \text{if } (i,j,k) \in \cup_{r \leq 0} S(\hat{q}, r), \\ 0_{\mathbb{F}}, & \text{otherwise.} \end{cases}$$

It is clear from the above definition that, given $\hat{q} \in \mathbb{Z}^n$, only finitely many of the $S(\hat{q}, r)$ are nonempty as r runs through $\mathbb{Z}$, and T is the disjoint union of these nonempty $S(\hat{q}, r)$.

Lemma. Let $\hat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$ and let $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$. Suppose further that $\lambda_{ijk} = 0_{\mathbb{F}}$ whenever $(i,j,k) \in \cup_{r<0} S(\hat{q}, r)$. Then $\boldsymbol{\lambda}(\hat{q}) \in \overline{O(\boldsymbol{\lambda})}$.

(In particular, the hypothesis of the lemma is satisfied regardless of the choice of $\boldsymbol{\lambda}$ by all $\hat{q} \in \mathbb{Z}^n$ such that $\cup_{r<0} S(\hat{q}, r) = \varnothing$.)

Example. Let $\hat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$ where $q_i = 1$, $1 \leq i \leq n$. Then $S(\hat{q}, r) = \varnothing$ for $r \neq 1$, so $\cup_{r<0} S(\hat{q}, r) = \varnothing$. Moreover, $\boldsymbol{\lambda}(\hat{q}) = \mathbf{0}$ (the structure vector of the abelian Lie algebra $\mathfrak{a}_n$) for any $\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$. It is now immediate from the above lemma that any (n-dimensional) algebra $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ degenerates to $\mathfrak{a}_n$ (a well-known result).

Various authors have considered necessary conditions for degenerations within special classes of algebras. Below we give some of these conditions in the general context of algebras over an arbitrary field.

Definition. Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$. Define the left annihilator of $\mathfrak{g}$ by $\operatorname{ann}_L \mathfrak{g} = \{c \in V : \ [c, a] = 0_V \text{ for all } a \in V\}$.

In particular, $\operatorname{ann}_L \mathfrak{g} = Z(\mathfrak{g})$, the center of $\mathfrak{g}$, when $\mathfrak{g} \in \mathcal{L}_n^a(\mathbb{F})$.

Considering certain $n \times n^2$ matrices formed by the structure constants of an algebra in $\mathcal{A}_n(\mathbb{F})$ we can prove the following result.

Lemma. Let $\mathfrak{g}, \mathfrak{g}_1 \in \mathcal{A}_n(\mathbb{F})$ and suppose that $\mathfrak{g}$ degenerates to $\mathfrak{g}_1$. Then, $\dim_{\mathbb{F}}(\operatorname{ann}_L \mathfrak{g}_1) \geq \dim_{\mathbb{F}}(\operatorname{ann}_L \mathfrak{g})$ and $\dim_{\mathbb{F}} \mathfrak{g}_1^2 \leq \dim_{\mathbb{F}} \mathfrak{g}^2$.

4 Orbit closures in $\mathcal{A}_n(\mathbb{F})$ consisting of precisely two orbits

First we introduce the algebra structures $\mathfrak{r}_n$ and $\mathfrak{h}_n \in \mathcal{L}_n^a(\mathbb{F})$.

Definition. Let $\boldsymbol{\rho} = (\rho_{ijk})$ and $\boldsymbol{\eta} = (\eta_{ijk}) \in \mathbb{F}^{n^3}$ be such that the only nonzero components of $\boldsymbol{\rho}$ are $\rho_{ini} = 1_{\mathbb{F}} = -\rho_{nii}$ for $1 \leq i \leq n-1$ and the only nonzero components of $\boldsymbol{\eta}$ are $\eta_{123} = 1_{\mathbb{F}} = -\eta_{213}$. It is easy to observe that $\boldsymbol{\rho}, \boldsymbol{\eta} \in \mathcal{L}_n(\mathbb{F})$. Now define $\mathfrak{r}_n, \mathfrak{h}_n \in \mathcal{L}_n^a(\mathbb{F})$ by $\mathfrak{r}_n = \Theta^{-1}(\boldsymbol{\rho})$ and $\mathfrak{h}_n = \Theta^{-1}(\boldsymbol{\eta})$ with Θ as defined in Section 2.

We denote by $\mathbb{F}\text{-sp}(x_1, \ldots, x_n)$ the set of $\mathbb{F}$-linear combinations of the elements $x_1, \ldots, x_n$ of V.

Definition. Let $\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{A}_n(\mathbb{F})$.

(i) We say that $\mathfrak{g}$ satisfies condition $(*)$ if $[x, y]_{\mathfrak{g}} \in \mathbb{F}\text{-sp}(x, y)$ for all $x, y \in V$.

(ii) We say that $\mathfrak{g}$ satisfies condition $(**)$ if $[x, x]_{\mathfrak{g}} \in \mathbb{F}\text{-sp}(x)$ for all $x \in V$.

It is then immediate that every $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ satisfies condition $(**)$. Moreover, $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ satisfies condition $(**)$ whenever $\mathfrak{g}$ satisfies condition $(*)$ but the converse is not true in general. It is easy to see that the subset $\{\boldsymbol{\lambda} \in \mathbb{F}^{n^3} : \boldsymbol{\lambda}$ is a structure vector for some $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ that satisfies condition $(*)\}$ of $\mathbb{F}^{n^3}$ is a union of orbits. Finally, we remark that $\mathfrak{h}_n$ does not satisfy condition $(*)$ whereas $\mathfrak{a}_n$, the n-dimensional abelian Lie algebra, satisfies condition $(*)$.

Lemma. Let $\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{A}_n(\mathbb{F})$ and suppose that $\mathfrak{g}$ satisfies condition $(**)$. Suppose further that $\mathfrak{g}$ does not satisfy condition $(*)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{h}_n$. (In particular, the hypothesis of the lemma is satisfied by any $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ which does not satisfy condition $(*)$.)

Next we introduce some more subsets of $\mathbb{F}^{n^3}$. Let $P = \{\boldsymbol{\lambda} = (\lambda_{ijk})_{1\leq i,j,k\leq n} \in \mathcal{K}_n(\mathbb{F}) : \lambda_{ijk} = 0_{\mathbb{F}}$ whenever $k \notin \{i,j\}$ and $\lambda_{iji} = \lambda_{kjk}$ whenever $j \notin \{i,k\}\}$. It is easy to see that P is an algebraic subset of $\mathbb{F}^{n^3}$. In fact, P is an n-dimensional $\mathbb{F}$-subspace of $\mathcal{K}_n(\mathbb{F})$. We also denote by $\boldsymbol{\rho}(\mathbb{F}G)$ and $\boldsymbol{\eta}(\mathbb{F}G)$ the $\mathbb{F}G$-submodules of $\mathbb{F}^{n^3}$ generated by $\boldsymbol{\rho}$ and $\boldsymbol{\eta}$ respectively.

Lemma. $P = O(\boldsymbol{\rho}) \cup \{\mathbf{0}\} = \overline{O(\boldsymbol{\rho})} = \boldsymbol{\rho}(\mathbb{F}G)$.

Corollary. (i) $\mathfrak{r}_n$ satisfies condition $(*)$.
(ii) Let $\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$. Then $\boldsymbol{\lambda}$ belongs to P if, and only if, $\boldsymbol{\lambda}$ is a structure vector for some algebra $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ which satisfies condition $(*)$.

Since the algebras $\mathfrak{r}_n$, $\mathfrak{h}_n$ are not isomorphic, we get that $P \cap O(\boldsymbol{\eta}) = \varnothing$. Moreover, combining the lemmas and corollary immediately above with the fact that $\mathbf{0} \in \overline{O(\boldsymbol{\lambda})}$ for any $\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$ we get

Corollary. Let $\boldsymbol{\lambda} \in \mathcal{K}_n(\mathbb{F}) \setminus P$. Then,
(i) $\boldsymbol{\eta} \in \overline{O(\boldsymbol{\lambda})}$.
(ii) If, in addition, $\boldsymbol{\lambda} \notin O(\boldsymbol{\eta})$, then $\overline{O(\boldsymbol{\lambda})}$ contains at least 3 distinct orbits.

Using the facts that $\dim_{\mathbb{F}} Z(\mathfrak{h}_n) = n - 2$ and that $\mathfrak{h}_n \in \mathcal{B}_n^a(\mathbb{F})$ we can prove the following two results.

Lemma. Let $\mathfrak{g} = (V, [,]) \in \mathcal{K}_n^a(\mathbb{F}) \cap \mathcal{B}_n^a(\mathbb{F})$ with $\dim_{\mathbb{F}}(\operatorname{ann} \mathfrak{g}) = n - 2$. Then $\mathfrak{g} \cong \mathfrak{h}_n$.

Lemma. The only proper degeneration of $\mathfrak{h}_n$ is to the abelian Lie algebra $\mathfrak{a}_n$. In other words, $\overline{O(\boldsymbol{\eta})} = O(\boldsymbol{\eta}) \cup \{\mathbf{0}\}$.

We can now state the first of our two main results. Its proof is the direct consequence of the above discussion.

Theorem. *Let $n \geq 3$ and let $\mathbb{F}$ be an arbitrary infinite field. Then, among all n-dimensional algebras satisfying the identity $[x, x] = 0$, algebras $\mathfrak{r}_n$ and $\mathfrak{h}_n$ are the only ones (up to isomorphism) which have the n-dimensional abelian Lie algebra $\mathfrak{a}_n$ as their only proper degeneration.*

Next we turn our attention to algebras in $\mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n(\mathbb{F})$ which have closures consisting of precisely two orbits.

Definition. We introduce the algebra structures $\mathfrak{d}_n$ and $\mathfrak{e}_n(\alpha)$, for $\alpha \in \mathbb{F}$, as follows. Let $\boldsymbol{\delta}_n = (\delta_{ijk}) \in \mathbb{F}^{n^3}$ be the structure vector which has $\delta_{112} = 1_{\mathbb{F}}$ as its only nonzero component. Also, for $\alpha \in \mathbb{F}$, let $\boldsymbol{\varepsilon}_n(\alpha) = (\varepsilon_{ijk}(\alpha)) \in \mathbb{F}^{n^3}$ be the structure vector which has $\varepsilon_{111}(\alpha) = 1_{\mathbb{F}}$, $\varepsilon_{1ii}(\alpha) = \alpha$ (for $2 \leq i \leq n$) and $\varepsilon_{i1i}(\alpha) = (1_{\mathbb{F}} - \alpha)$ (for $2 \leq i \leq n$) as its only components which can possibly be nonzero. Finally define the algebra structures $\mathfrak{d}_n, \mathfrak{e}_n(\alpha) \in \mathcal{A}_n(\mathbb{F})$ by $\mathfrak{d}_n = \Theta^{-1}(\boldsymbol{\delta}_n)$ and $\mathfrak{e}_n(\alpha) = \Theta^{-1}(\boldsymbol{\varepsilon}_n(\alpha))$ with Θ as in Section 2.

The proof of the second of our two main results relies on the following two lemmas.

Lemma. Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$ and suppose $\mathfrak{g}$ does not satisfy condition $(**)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{d}_n$.

Lemma. Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n^a(\mathbb{F})$ and suppose $\mathfrak{g}$ satisfies condition $(*)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{e}_n(\alpha)$ for some $\alpha \in \mathbb{F}$.

A key step in the proof of the above lemmas is to pick a suitable "starting" basis for the algebras involved.

Theorem. *Let $n \geq 3$ and let $\mathbb{F}$ be an arbitrary infinite field. Then $\mathfrak{d}_n$ together with the family $\{\mathfrak{e}_n(\alpha) : \alpha \in \mathbb{F}\}$ give a complete list of non-isomorphic elements of $\mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n^a(\mathbb{F})$ which have $\mathfrak{a}_n$ as their only proper degeneration.*

5 On the composition series of the $\mathbb{F}G$-module $\mathcal{K}_n(\mathbb{F})$

Definition. Let $\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{A}_n(\mathbb{F})$. Fix $x \in V$. We define the adjoint map in $\mathfrak{g}$ (relative to x) by $\mathrm{ad}_x : V \to V : y \mapsto [x, y]_{\mathfrak{g}}$, $(y \in V)$. Then ad_x is an $\mathbb{F}$-linear map. We say that the algebra structure $\mathfrak{g}$ is unimodular if $\mathrm{trace}(\mathrm{ad}_x) = 0_{\mathbb{F}}$ for each $x \in V$.

Definition. Define $\mathcal{U}_n(\mathbb{F}) = \{\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathcal{K}_n(\mathbb{F}) : \sum_{j=1}^{n} \lambda_{ijj} = 0_{\mathbb{F}} \text{ for } i = 1, \ldots, n\}$. This is the set of structure vectors corresponding to unimodular algebra structures in $\mathcal{K}_n^a(\mathbb{F})$.

It is easy to see that $\mathcal{U}_n(\mathbb{F})$ is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$ and that $\mathcal{U}_n(\mathbb{F})$ is also a union of orbits. We can thus regard $\mathcal{U}_n(\mathbb{F})$ as an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$.

Lemma. (i) $\mathcal{U}_n(\mathbb{F}) = \boldsymbol{\eta}(\mathbb{F}G)$.
(ii) $\boldsymbol{\rho} \in \mathcal{U}_n(\mathbb{F})$ if, and only if, $\mathrm{char}\,\mathbb{F} | (n-1)$.

Below we discuss briefly certain observations on the composition series of $\mathcal{K}_n(\mathbb{F})$ as an $\mathbb{F}G$-module. For this we need to consider the cases $\mathrm{char}\,\mathbb{F} \nmid (n-1)$ and $\mathrm{char}\,\mathbb{F} | (n-1)$ separately.

(i) case $\operatorname{char}\mathbb{F} \not| (n-1)$: Then $\boldsymbol{\rho}(\mathbb{F}G) \cap \boldsymbol{\eta}(\mathbb{F}G) = \{\mathbf{0}\}$ in view of the above remark (recall $\boldsymbol{\eta}(\mathbb{F}G) = \mathcal{U}_n(\mathbb{F})$ always). Moreover, our assumption on $\operatorname{char}\mathbb{F}$ ensures that $\boldsymbol{\eta}(\mathbb{F}G)$ is an irreducible $\mathbb{F}G$-module. So, in this case, $\boldsymbol{\rho}(\mathbb{F}G)$ and $\boldsymbol{\eta}(\mathbb{F}G)$ are both irreducible $\mathbb{F}G$-submodules of $\mathcal{K}_n(\mathbb{F})$. The results of the previous section now ensure that $\mathcal{K}_n(\mathbb{F})$ has precisely two composition series, namely $\{\mathbf{0}\} \subseteq \boldsymbol{\rho}(\mathbb{F}G) \subseteq \mathcal{K}_n(\mathbb{F})$ and $\{\mathbf{0}\} \subseteq \boldsymbol{\eta}(\mathbb{F}G) \subseteq \mathcal{K}_n(\mathbb{F})$.

(ii) case $\operatorname{char}\mathbb{F} | (n-1)$: Then $\boldsymbol{\rho} \in \boldsymbol{\eta}(\mathbb{F}G)$ and hence $\boldsymbol{\rho}(\mathbb{F}G) \subseteq \boldsymbol{\eta}(\mathbb{F}G)$. Similar argument as above then shows that every composition series for $\mathcal{K}_n(\mathbb{F})$ necessarily begins with $\{\mathbf{0}\} \subseteq \boldsymbol{\rho}(\mathbb{F}G) \subseteq \boldsymbol{\eta}(\mathbb{F}G)$.