Set-theoretical solutions of the pentagon equation on groups

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Solutions of the pentagon equation on a vector space

Definition

Let \( V \) be a vector space over a field \( K \). A linear operator \( S \in \text{End}(V \otimes V) \) is called a **solution of the pentagon equation (PE)** if it satisfies

\[
S_{12} S_{13} S_{23} = S_{23} S_{12},
\]

where \( S_{12} = S \otimes \text{id}_V \), \( S_{23} = \text{id}_V \otimes S \), \( S_{13} = (\tau \otimes \text{id}_V) S_{12} (\tau \otimes \text{id}_V) \) (here \( \tau \) denotes the flip map \( v \otimes w \rightarrow w \otimes v \)).
Set-theoretical solutions of the pentagon equation

Definition

A set-theoretical solution of the pentagon equation (PE) on an arbitrary set $M$ is a map $s : M \times M \rightarrow M \times M$ which satisfies the "reversed" pentagon relation

$$s_{23} s_{13} s_{12} = s_{12} s_{23},$$

where $s_{12} = s \times \text{id}_M$, $s_{23} = \text{id}_M \times s$ and $s_{13} = (\text{id}_M \times \tau) s_{12} (\text{id}_M \times \tau)$ (here $\tau$ denotes the flip map $(x, y) \rightarrow (y, x)$).
Link between the two equations

Let $K$ be a field, $M$ a finite set, and $V := K^M$. Then, $V \otimes V \cong K^{M \times M}$.

For each map $s : M \times M \to M \times M$ one can associate its pull-back $S$, i.e., the linear operator $S \in \text{End}(V \otimes V)$ such that

$$(S \varphi)(x, y) = \varphi(s(x, y)), \quad \varphi \in K^{M \times M}.$$

$S$ is a solution of the PE if and only if $s$ is a set-theoretical solution of the PE.

Hereinafter, we briefly call a set-theoretical solution $s$ a solution.
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Hereinafter, we briefly call a set-theoretical solution $s$ a *solution*.
A pioneering work

For a map \( s : M \times M \to M \times M \) define binary operations \( \cdot \) and \( * \) via
\[
s(x, y) = (x \cdot y, x * y),
\]
for all \( x, y \in M \).

**Proposition (Kashaev-Sergeev, 1998)**

The map \( s \) is a solution on \( M \) if and only if the following conditions hold

1. \( (x \cdot y) \cdot z = x \cdot (y \cdot z) \),
2. \( (x * y) \cdot ((x \cdot y) * z) = x * (y \cdot z) \),
3. \( (x * y) \cdot ((x \cdot y) \cdot z) = y \cdot z \),

for all \( x, y, z \in M \).
A pioneering work

For a map $s : M \times M \to M \times M$ define binary operations $\cdot$ and $\ast$ via

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**Proposition (Kashaev-Sergeev, 1998)**

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1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
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Examples of solutions

1. *(Militaru solutions)* If $M$ is a set, $f, g$ are maps from $M$ into itself such that $f^2 = f$, $g^2 = g$, and $fg = gf$, then

   $$s(x, y) = (f(x), g(y))$$

   is a solution on $M$.

2. If $(M, \cdot)$ is a semigroup, $\gamma \in \text{End}(M)$, $\gamma^2 = \gamma$, the map

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3. *(Kac-Takesaki solutions)* If $(M, \cdot)$ is a group, the maps

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Examples of solutions on factorizable groups

Let \( M \) be a group and \( A, B \) two subgroups such that \( A \cap B = \{1\} \) and \( M = AB \). Let \( p_1 : M \to A \) and \( p_2 : M \to B \) be maps such that \( x = p_1(x) \, p_2(x) \), for every \( x \in M \).

4. (Zakrzewski solution) The map

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s(x, y) = (p_2(yp_1(x)^{-1}) \, x, \, yp_1(x)^{-1})
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A question arises

Kashaev and Sergeev proved that if \((M, \cdot)\) is a group, then the only invertible solution \(s(x, y) = (x \cdot y, x \ast y)\) on \(M\) is given by

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Question

Are there any other solutions of the form \(s(x, y) = (x \cdot y, x \ast y)\) on a group \((M, \cdot)\), if \(s\) is not invertible?
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A new notation

If \( s(x, y) = (x \cdot y, x \ast y) \) is a solution, we set \( x \ast y =: \theta_x(y) \), for all \( x, y \in M \), where \( \theta_x : M \to M \) is a map, for every \( x \in M \).

**Proposition**

*The map \( s(x, y) = (x \cdot y, \theta_x(y)) \) is a solution on a set \( M \) if and only if*

1. \( (M, \cdot) \) is a semigroup,
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The kernel of a solution

Although $\theta_1$ is not a homomorphism, we have the following result.

**Proposition (Catino, Miccoli, M., 2019)**

Let $s(x, y) = (x \cdot y, \theta_x(y))$ be a solution on a group $(M, \cdot)$. The subset of $M$

$$K := \{ x \mid x \in M, \theta_1(x) = 1 \},$$

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Description of solutions on groups

Theorem (Catino, Miccoli, M., 2019)

Let \((M, \cdot)\) be a group and \(K \trianglelefteq M\). Moreover, consider

- \(R\) a system of representatives of \(M/K\) such that \(1 \in R\),
- \(\mu : M \to R\) a map such that \(\mu(x) \in K \cdot x\), for every \(x \in M\).

Then, the map \(s : M \times M \to M \times M\) given by

\[
s(x, y) = (x \cdot y, \mu(x)^{-1} \cdot \mu(x \cdot y)),
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for all \(x, y \in M\), is a solution on \(M\).

Conversely, if \(s(x, y) = (x \cdot y, \theta_x(y))\) is a solution on \(M\), for all \(x, y \in M\), there exists \(K \trianglelefteq M\), the kernel of \(s\), such that

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Applications

1. The unique invertible solution on a group \((M, \cdot)\) is given by

\[
s(x, y) = (x \cdot y, y).
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2. Let \(n \geq 3\) and \(S_n\) the symmetric group of order \(n\). Consider

- \(K = A_n\),
- \(R = \{\text{id}_{S_n}, \pi\}\), where \(\pi\) is a transposition of \(S_n\);
- the map \(\mu : S_n \to R\) given by

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\mu(\alpha) = \begin{cases} 
\pi & \text{if } \alpha \text{ is odd} \\
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Solutions on groups \((M, \ast)\)

Another question is to describe all the solutions \(s(x, y) = (x \cdot y, x \ast y)\) when \((M, \ast)\) is a group.

Proposition (Catino, Miccoli, M., 2019)

Let \((M, \ast)\) be a group. Then, \(s(x, y) = (x \cdot y, x \ast y)\) is a solution on \(M\) if and only if

- \((M, \ast)\) is an elementary abelian 2-group,
- \(x \cdot y = x\) holds, for all \(x, y \in M\).
Solutions on groups \((M, \ast)\)

Another question is to describe all the solutions \(s(x, y) = (x \cdot y, x \ast y)\) when \((M, \ast)\) is a group.

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How to find new examples of solutions on semigroups \((S, \cdot)\)?

In order to obtain new solutions, we focus on constructions of solutions on the Cartesian product of two semigroups \(S\) and \(T\) [Catino, Stefanelli, M., in preparation].

From now on, we set \(x \cdot y =: xy\).
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Matched product of solutions - I

Let \( S, T \) be semigroups, \( s(a, b) = (ab, \theta_a(b)) \) and \( t(u, v) = (uv, \theta_u(v)) \) solutions on \( S \) and \( T \), respectively. Let \( \alpha : T \to S^S \) and \( \beta : S \to T^T \) be two maps, and set

\[
\forall u \in T \quad \alpha_u := \alpha(u), \quad \forall a \in S \quad \beta_a := \beta(a).
\]

If the following conditions are satisfied

\[
\begin{align*}
\alpha_v (a \alpha_u (b)) &= \alpha_v (a) \alpha_{\beta_a(v)u} (b), \\
\beta_c (\beta_b (u)v) &= \beta_{b \alpha_v(c)} (u) \beta_c (v), \\
\theta_{a \alpha_u (b)} &= \alpha \theta_{\beta_b (u) \beta_v(c)}(v) \theta_{a \alpha_u (b)}, \\
\theta_{a \alpha_u} &= \theta_{a \alpha_v(a)} \alpha_{\beta_a(v)u}, \\
\beta_{\theta_{a \alpha_u (b) \beta_b (u)v(c)} \theta_{\beta_b (u) \beta_v(c)}}(v) &= \theta_{\beta_{b \alpha_v(c)}(u) \beta_c (v)},
\end{align*}
\]

for all \( a, b, c \in S \) and \( u, v \in T \), then we call \((s, t, \alpha, \beta)\) a matched quadruple.
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Let $S$, $T$ be semigroups, $s(a, b) = (ab, \theta_a(b))$ and $t(u, v) = (uv, \theta_u(v))$ solutions on $S$ and $T$, respectively. Let $\alpha : T \to S^S$ and $\beta : S \to T^T$ be two maps, and set

$$\forall u \in T \quad \alpha(u) := \alpha(u), \quad \forall a \in S \quad \beta(a) := \beta(a).$$

If the following conditions are satisfied

$$\alpha_v(a\alpha_u(b)) = \alpha_v(a)\beta_{\alpha_u(b)},$$
$$\beta_c(\beta_b(u)v) = \beta_{\beta_b(u)}(u)\beta_c(v),$$
$$\theta_{a\alpha_u(b)} = \alpha_{\theta_{\beta_b(u)}(v)}\theta_{a\alpha_u(b)},$$
$$\theta_{a\alpha_u(b)} = \theta_{\alpha_v(a)}\beta_{\alpha_v(a)},$$
$$\beta_{\theta_{a\alpha_u(b)}\beta_{\beta_b(u)v}}(v) = \beta_{\theta_{\beta_b(u)}(v)}\beta_c(v),$$

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$$\theta_{a\alpha_u(b)} = \alpha_{\theta_{\beta_b(u)v}}(v)\theta_{a\alpha_u(b)},$$
$$\theta_a\alpha_u = \theta_{\alpha_v(a)}\alpha_{\beta_a(v)u},$$
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$$\theta_a \alpha_u = \theta_{\alpha_v(a)} \alpha_{\beta_a(v)} u,$$
$$\beta_{\theta_{a \alpha_u(b)} \alpha_{\beta_b(u)} v} (\theta_{\beta_b(u)}(v)) = \theta_{\beta_{\alpha_u(c)}(u)} \beta_c (v),$$

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for all $a, b, c \in S$ and $u, v \in T$, then we call $(s, t, \alpha, \beta)$ a matched quadruple.
Matched product of solutions - II

The first two conditions

\[ \alpha_v (a\alpha_u (b)) = \alpha_v (a) \alpha_{\beta_a(v)u} (b), \]
\[ \beta_c (\beta_b (u) v) = \beta_{b\alpha_v(c)} (u) \beta_c (v) \]

ensure that \( S \times T \) endowed with the operation defined by

\[ (a, u) (b, v) = (a\alpha_u(b), \beta_b(u)v), \]

is a semigroup, called the **matched product of \( S \) and \( T \)**, and we denote it by \( S \bowtie T \).

**Theorem (Catino, Stefanelli, M., work in progress)**

Let \( S \) and \( T \) be two semigroups, and \( (s, t, \alpha, \beta) \) a matched quadruple. Then, the map \( s \bowtie t : (S \times T)^2 \to (S \times T)^2 \) defined by

\[ s \bowtie t (a, u ; b, v) = (a\alpha_u(b), \beta_b(u)v ; \theta_a\alpha_u(b), \theta_{\beta_b(u)}(v)) , \]

for all \( (a, u), (b, v) \in S \times T \), is a solution on \( S \bowtie T \).
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\[ \alpha_v (a \alpha_u (b)) = \alpha_v (a) \alpha_{\beta_a(v)u} (b), \]
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is a semigroup, called the \textit{matched product of S and T}, and we denote it by \( S \Join T \).

**Theorem (Catino, Stefanelli, M. , work in progress)**

\textit{Let S and T be two semigroups, and \((s, t, \alpha, \beta)\) a matched quadruple. Then, the map \( s \Join t : (S \times T)^2 \rightarrow (S \times T)^2 \) defined by}

\[ s \Join t (a, u; b, v) = (a \alpha_u (b), \beta_b (u) v; \theta_a \alpha_u (b), \theta_{\beta_b(u)} (v)), \]

\textit{for all \((a, u), (b, v) \in S \times T\), is a solution on \( S \Join T \).}
Matched product of solutions - II

The first two conditions

\[ \alpha_v (a\alpha_u (b)) = \alpha_v (a) \alpha_{\beta_a (v)u} (b), \]
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for all \((a, u), (b, v) \in S \times T\), is a solution on \( S \Join T \).
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for all \((a, u), (b, v) \in S \times T\), is a solution on \( S \bowtie T \).
Remark

In addition, if $S$ and $T$ are monoids with identity $1_S$ and $1_T$ respectively, we have to require the following conditions

\begin{align*}
\alpha_1 &= \text{id}_S, \\
\beta_1 &= \text{id}_T, \\
\forall \ a \in S \quad &\beta_a(1_T) = 1_T, \\
\forall \ u \in T \quad &\alpha_u(1_S) = 1_S,
\end{align*}

so that $S \bowtie T$ is a monoid with identity $(1_S, 1_T)$.

In this case, if $(s, t, \alpha, \beta)$ is a matched quadruple, conditions become easier:

\begin{align*}
\theta_s &= \alpha_{\theta_u(v)}\theta_s = \theta_{\alpha_v(a)}\beta_a(v), \\
\beta_{\theta_a\alpha_{uv}(b)}\theta_u(v) &= \theta_{\beta_{\alpha_v(b)}(u)}\beta_b(v).
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Remark

In addition, if $S$ and $T$ are monoids with identity $1_S$ and $1_T$ respectively, we have to require the following conditions

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\alpha_{1_T} = \text{id}_S, \\
\beta_{1_S} = \text{id}_T, \\
\forall a \in S \quad \beta_a(1_T) = 1_T, \\
\forall u \in T \quad \alpha_u(1_S) = 1_S,
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In this case, if $(s, t, \alpha, \beta)$ is a matched quadruple, conditions become easier:

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\theta_a = \alpha_{\theta_u(v)} \theta_a = \theta_{\alpha_v(a)} \alpha_{\beta_a(v)}, \\
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\]
An example

Let us consider

- $S$ a semigroup, $\gamma \in \text{End}(S)$, $\gamma^2 = \gamma$, and $s(a, b) = (ab, \gamma(b))$ a solution on $S$;

- $T$ a semigroup and $t(u, v) = (uv, v)$ a solution on $T$;

- $\alpha_u = \gamma$, for every $u \in T$;

- $\beta_a = \text{id}_T$, for every $a \in S$.

Then, $(s, t, \alpha, \beta)$ is a matched quadruple and so the map

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M. Mazzotta | Set-theoretical solutions of the pentagon equation on groups
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A link between the pentagon equation and the Yang-Baxter equation

Tomorrow at Paola Stefanelli’s talk
Thanks for your attention!