Groups with all Subgroups Permutable or Soluble of Finite Rank

Maria Ferrara
Università degli Studi di Napoli Federico II
Introduction

The theory of groups all of whose subgroups satisfy some property has a very long history, beginning in 1897 with R. Dedekind who studied groups all of whose subgroups are normal, the so-called Dedekind groups.

Richard Dedekind

The theory of groups all of whose subgroups satisfy some property has a very long history, beginning in 1897 with R. Dedekind (1831 – 1916) who studied groups all of whose subgroups are normal, the so-called Dedekind groups.

Obviously all abelian groups are Dedekind groups and a non-abelian Dedekind group is called Hamiltonian group.
The smallest example of a Hamiltonian group is the quaternion group of order 8, denoted by $Q_8$ and discovered by William R. Hamilton in 1843.

“Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ & cut it on a stone of this bridge”
However, every Hamiltonian group contains a copy of $Q_8$, in fact and a classical result of R. Baer and R. Dedekind (in the infinite and respectively finite order case)...

Reinhold Baer


(1902 – 1979)
\dots shows that every Hamiltonian group is a direct product of the form

\[ G = Q_8 \times B \times D \]
Introduction

shows that every Hamiltonian group is a direct product of the form

\[ G = Q_8 \times B \times D \]

where

- \( B \) is an elementary abelian 2-group
shows that every Hamiltonian group is a direct product of the form

\[ G = Q_8 \times B \times D \]

and

- \( D \) is a periodic abelian group with all elements of odd order
Normal subgroups have several useful generalizations, one of which is the permutability.
Normal subgroups have several useful generalizations, one of which is the permutability.

A subgroup $H$ of a group $G$ is said to be \textit{permutable} in $G$ if
Normal subgroups have several useful generalizations, one of which is the permutability.

A subgroup $H$ of a group $G$ is said to be \textit{permutable} in $G$ if

$$HK = KH$$
Normal subgroups have several useful generalizations, one of which is the **permutability**.

A subgroup $H$ of a group $G$ is said to be **permutable** in $G$ if

$$HK = KH$$

for every subgroup $K$ of $G$. 

Maria Ferrara | Groups with all subgroups permutable or soluble of finite rank
These subgroups had been introduced by Øystein Ore in 1937, who called them “quasinormal”.

Øystein Ore

“Structures and group theory”,
Obviously, normal subgroups are always permutable, but the converse is not true.
Obviously, normal subgroups are always permutable, but there are numerous example of non-normal permutable subgroups.

Example
Let $p$ be an odd prime, and let $G$ be an extraspecial group of order $p^3$ and exponent $p^2$. 


Introduction

Obviously, normal subgroups are always permutable, but there are numerous example of non-normal permutable subgroups.

Example
Let $p$ be an odd prime, and let $G$ be an extraspecial group of order $p^3$ and exponent $p^2$.

- $G$ has all subgroups permutable
Introduction

Obviously, normal subgroups are always permutable, but there are numerous example of non-normal permutable subgroups.

Example
Let $p$ be an odd prime, and let $G$ be an extraspecial group of order $p^3$ and exponent $p^2$.

- $G$ has all subgroups permutable
- $G$ has non-normal subgroups
Groups with all subgroups permutable are called quasi-Hamiltonian (or quasi-Abelian) groups and they were studied by Kenkichi Iwasawa who classified them.

Kenkichi Iwasawa

“Einege sätze über freie gruppen”,

(1917–1998)
What about the structure of groups in which the set of all non-permutable subgroups is “small” in some sense?
I examined this problem together with Martyn R. Dixon
I examined this problem together with Martyn R. Dixon, Z. Yalcin Karatas
I examined this problem together with Martyn R. Dixon, Zekeriya Yalcin Karatas and Marco Trombetti
I examined this problem together with Martyn R. Dixon, Zekeriya Yalcin Karatas and Marco Trombetti in the wake of the following paper

S. Franciosi, F. de Giovanni and M. L. Newell
“Groups with Polycyclic Non-Normal Subgroups”,
Algebra Colloq. 7 (2000), no. 1, 33-42.
Dually to their results for the non-normal subgroups, we have proved that

- If $G$ is a group whose non-permutable subgroups are periodic, then $G$ is quasihamiltonian or periodic.
Dually to their results for the non-normal subgroups, we have proved that

- If $G$ is a group whose non-permutable subgroups are periodic, then $G$ is quasihamiltonian or periodic.

- If $G$ is locally graded group and the non-permutable subgroups of $G$ are locally finite, then $G$ is quasihamiltonian or locally finite.
Preliminaries

Dually to their results for the non-normal subgroups, we have proved that

- If $G$ is a group whose non-permutable subgroups are periodic, then $G$ is quasihamiltonian or periodic

- If $G$ is locally graded group and the non-permutable subgroups of $G$ are locally finite, then $G$ is quasihamiltonian or locally finite

- If $G$ a locally graded group whose subgroups are permutable or Černikov, then $G$ is quasihamiltonian or Černikov
If $G$ is locally graded group and the non-permutable subgroups of $G$ are locally finite, then $G$ is quasihamiltonian or locally finite.

If $G$ a locally graded group whose subgroups are permutable or Černikov, then $G$ is quasihamiltonian or Černikov.

Why a locally graded group?
If $G$ is locally graded group and the non-permutable subgroups of $G$ are locally finite, then $G$ is quasihamiltonian or locally finite.

If $G$ a locally graded group whose subgroups are permutable or Černikov, then $G$ is quasihamiltonian or Černikov.

A group $G$ is said to be locally graded if all its finitely generated nontrivial subgroups have a nontrivial finite image and the consideration of Tarski groups shows that a requirement of this type is necessary for this kind of problem.

Why a locally graded group?
In 1967, S.N. Černikov showed that an infinite locally graded group whose infinite subgroups are normal.
In 1967, S.N. Černikov showed that an infinite locally graded group whose infinite subgroups are normal is either a Dedekind group.

S.N. Černikov


(1912 – 1987)
In 1967, S.N. Černikov showed that an infinite locally graded group whose infinite subgroups are normal is either a Dedekind group or an extension of a Prüfer group by a finite Dedekind group.

S.N. Černikov


(1912 – 1987)
Generalizing this result, we have proved that in an infinite locally graded group $G$, every infinite subgroup is permutable if and only if $G$ is quasihamiltonian.
Generalizing this result, we have proved that in an infinite locally graded group $G$,

every infinite subgroup is permutable if and only if $G$ is quasihamiltonian or $G$ is an extension of a Prüfer group by a finite quasihamiltonian group.
Ok! But, it didn’t occur to you it might exist further generalizations?
Ok! But, it didn’t occur to you it might exist further generalizations?

Of course!
The first more general property we consider is the finiteness for abelian section rank.
The first more general property we consider is the finiteness for abelian section rank.

A group $G$ has **finite abelian section rank** if every abelian section of $G$ has finite 0-rank and finite $p$-rank for all primes $p$. 
Let $G$ be a locally graded group. Then all subgroups are permutable or soluble of finite abelian section rank if and only if either
Let $G$ be a locally graded group. Then all subgroups are permutable or soluble of finite abelian section rank if and only if either

1. $G$ is quasihamiltonian;
Let $G$ be a locally graded group. Then all subgroups are permutable or soluble of finite abelian section rank if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is soluble of finite abelian section rank;
Let $G$ be a locally graded group. Then all subgroups are permutable or soluble of finite abelian section rank if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is soluble of finite abelian section rank;

3. $G$ has finite abelian section rank and $G''$ is a finite perfect minimal non-soluble group such that $G/G''$ is quasihamiltonian.
Finite rank

A group $G$ is said to have finite (Prüfer) rank $r = r(G)$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with such property.
As a corollary, we have that in a locally graded group $G$, all subgroups are permutable or soluble of finite rank if and only if either
As a corollary, we have that in a locally graded group $G$, all subgroups are permutable or soluble of finite rank if and only if either

1. $G$ is quasihamiltonian;
As a corollary, we have that in a locally graded group $G$, all subgroups are permutable or soluble of finite rank if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is soluble of finite rank;
As a corollary, we have that in a locally graded group $G$, all subgroups are permutable or soluble of finite rank if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is soluble of finite rank;

3. $G$ has finite rank and $G''$ is a finite perfect minimal non-soluble group such that $G/G''$ is quasihamiltonian.
Do you also know what happens in a $S_1$-group?
A group $G$ is called an $S_1$-group if $G$ is a hyperabelian group with finite abelian section rank and $G$ contains elements of only finitely many distinct prime orders.

Thus, in particular, $S_1$-groups are certain types of soluble group with finite special rank.
Let $G$ be a locally graded group. Then all subgroups are permutable or $S_1$ if and only if either
Let $G$ be a locally graded group. Then all subgroups are permutable or $S_1$ if and only if either

1. $G$ is quasihamiltonian;
Let $G$ be a locally graded group. Then all subgroups are permutable or $S_1$ if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is an $S_1$-group;
Let $G$ be a locally graded group. Then all subgroups are permutable or $S_1$ if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is an $S_1$-group;

3. $G$ is a finite extension of an $S_1$-group and $G''$ is a finite perfect minimal non-soluble group such that $G/G''$ is quasihamiltonian.
Next we recall that a group $G$ is called \textit{minimax} if it has a finite subnormal series whose factors either satisfy \textit{min} or \textit{max}.
Next we recall that a group $G$ is called \textit{minimax} if it has a finite subnormal series whose factors either satisfy \textit{min} or \textit{max}.
Next we recall that a group $G$ is called **minimax** if it has a finite subnormal series whose factors either satisfy min or max.

Clearly every soluble minimax group is an $S_1$-group.
Let $G$ be a locally graded group. Then all subgroups are permutable or soluble minimax if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is a soluble minimax group;

3. $G$ is a minimax group and $G''$ is a finite perfect minimal non-soluble group such that $G/G''$ is quasihamiltonian.
Minimax groups

Let $G$ be a locally graded group. Then all subgroups are permutable or soluble minimax if and only if either

1. $G$ is quasihamiltonian;

2. $G$ is a soluble minimax group;

3. $G$ is a minimax group and $G''$ is a finite perfect minimal non-soluble group such that $G/G''$ is quasihamiltonian.
Let $G$ a soluble minimax group all of whose non-permutable subgroups are polycyclic.
Minimax groups

Let $G$ a soluble minimax group all of whose non-permutable subgroups are polycyclic.

If the finite residual $R$ of $G$ is a direct product of at least two Prüfer subgroups, then $G$ is quasihamiltonian.
Periodic case

Periodic groups of the type we are interested in are now easy to describe.
Periodic case

Periodic groups of the type we are interested in are now easy to describe.

Let $G$ be an infinite non-quasihamiltonian periodic soluble minimax group and let $J$ be the finite residual of $G$. 
Periodic case

**Periodic groups** of the type we are interested in are now easy to describe.

Let $G$ be an infinite non-quasihamiltonian **periodic soluble minimax** group and let $J$ be the finite residual of $G$.

Then all non-permutable subgroups of $G$ are polycyclic **if and only if** $J$ is a Prüfer $p$-group for some prime $p$ and $G/J$ is a finite quasihamiltonian group.
Non-periodic case

Let $G$ be a non-periodic soluble minimax group with nontrivial finite residual $J$ and suppose that $G$ is not quasihamiltonian. If all non-permutable subgroups of $G$ are polycyclic, then the following conditions hold:

1. $J$ is a Prüfer $p$-group for some prime $p$.
2. $G/J$ is quasihamiltonian and the set $T$ of elements of finite order in $G$ is a subgroup such that $G/T$ is torsion-free abelian.
3. $T/J$ is a finite abelian group.

Furthermore, either

4. Every abelian subgroup is min-by-max, or
5. $G$ contains an infinitely generated torsion-free abelian subgroup.

Conversely, if $G$ satisfies 1-4, then all non-permutable subgroups of $G$ are polycyclic.
Non-periodic case

Let $G$ be a non-periodic soluble minimax group with nontrivial finite residual $J$ and suppose that $G$ is not quasihamiltonian. If all non-permutable subgroups of $G$ are polycyclic, then the following conditions hold:

1. $J$ is a Prüfer $p$-group for some prime $p$
2. $G/J$ is quasihamiltonian and the set $T$ of elements of finite order in $G$ is a subgroup such that $G/T$ is torsion-free abelian
3. $T/J$ is a finite abelian group
Non-periodic case

Let $G$ be a non-periodic soluble minimax group with nontrivial finite residual $J$ and suppose that $G$ is not quasihamiltonian.
If all non-permutable subgroups of $G$ are polycyclic, then the following conditions hold:

1. $J$ is a Prüfer $p$-group for some prime $p$
2. $G/J$ is quasihamiltonian and the set $T$ of elements of finite order in $G$ is a subgroup such that $G/T$ is torsion-free abelian
3. $T/J$ is a finite abelian group

Furthermore either
4. Every abelian subgroup is min-by-max, or
5. $G$ contains an infinitely generated torsion-free abelian subgroup.
Non-periodic case

Let $G$ be a non-periodic soluble minimax group with nontrivial finite residual $J$ and suppose that $G$ is not quasihamiltonian. If all non-permutable subgroups of $G$ are polycyclic, then the following conditions hold:

1. $J$ is a Prüfer $p$-group for some prime $p$
2. $G/J$ is quasihamiltonian and the set $T$ of elements of finite order in $G$ is a subgroup such that $G/T$ is torsion-free abelian
3. $T/J$ is a finite abelian group

Furthermore either

4. Every abelian subgroup is min-by-max, or
5. $G$ contains an infinitely generated torsion-free abelian subgroup.

Conversely, if $G$ satisfies 1-4, then all non-permutable subgroups of $G$ are polycyclic.
Non-periodic case

Let $G$ be a non-periodic soluble minimax group with nontrivial finite residual $J$ and suppose that $G$ is not quasihamiltonian. If all non-permutable subgroups of $G$ are polycyclic, then the following conditions hold:

1. $J$ is a Prüfer $p$-group for some prime $p$
2. $G/J$ is quasihamiltonian and the set $T$ of elements of finite order in $G$ is a subgroup such that $G/T$ is torsion-free abelian
3. $T/J$ is a finite abelian group

Furthermore either
4. Every abelian subgroup is min-by-max, or
5. $G$ contains an infinitely generated torsion-free abelian subgroup.

Conversely, if $G$ satisfies 1-4, then all non-permutable subgroups of $G$ are polycyclic.
Non-periodic case

Let $G$ be a non-periodic, residually finite soluble minimax group that is neither polycyclic nor quasihamiltonian.
Let $G$ be a non-periodic, residually finite soluble minimax group that is neither polycyclic nor quasihamiltonian.

If all non-permutable subgroups of $G$ are polycyclic, then $G$ is nilpotent and central-by-finite and splits over the torsion part.
Thank you for your attention!