Varieties of superalgebras with superinvolution

Antonio Ioppolo

University of Palermo
Department of Mathematics and Computer Science

Advances in Group Theory and Applications
September 5-8, 2017, Lecce

1. Preliminaries
   - Algebras with polynomial identities
   - Superalgebras with superinvolution
2. A characterization of $*$-varieties of polynomial growth
   - Varieties of almost polynomial growth
   - The analogous of Kemer's theorem
3. Classification of the subvarieties
   - Subvarieties of $\mathrm{var}^*(F \oplus F)$
   - Subvarieties of $\mathrm{var}^*(M)$
   - Subvarieties of $\mathrm{var}^*(M^{sup})$
   - Subvarieties of $\mathrm{var}^*(G^\sharp)$ and $\mathrm{var}^*(G^\star)$

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Let A be an associative F-algebra. Then $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity of A, and we write $f \equiv 0$, if, for all $a_1, \ldots, a_n \in A$, $f(a_1, \ldots, a_n) = 0$.

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Example

Any commutative algebra C is a PI-algebra since $[x_1, x_2] \equiv 0$ on C.

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Theorem (Regev, 1972)

For any PI-algebra A, the codimension sequence $c_n(A)$, $n = 1, 2, \ldots,$ is exponentially bounded.

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A variety $\mathcal{V}$ has

- polynomial growth if $c_n(\mathcal{V})$ is polynomially bounded;
- almost polynomial growth if $c_n(\mathcal{V})$, $n = 1, 2, \ldots$, is not polynomially bounded but any proper subvariety of $\mathcal{V}$ has polynomial growth.

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Theorem (Kemer, 1979)

A variety $\mathcal{V}$ has polynomial growth if and only if G, $UT_2 \notin \mathcal{V}$.

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Given an n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$, it is possible to define a $\mathbb{Z}_2$-grading on $M_n(F)$, called elementary, by setting

$$M_n(F)_i = span_F \{e_{ij} \mid g_i + g_j = i\}, \quad i = 0, 1.$$

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The free algebra with superinvolution $F\langle Y \cup Z, *\rangle$ is generated by symmetric and skew elements of homogeneous degree 0 and 1,

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Let $f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_t^+, z_1^-, \dots, z_s^-)$ a $*$-polynomial of $F\langle Y \cup Z, *\rangle$. We say that f is a $*$-identity of $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ if, for all $u_1^+, \dots, u_n^+ \in A_0^+$, $u_1^-, \dots, u_m^- \in A_0^-$, $v_1^+, \dots, v_t^+ \in A_1^+$ e $v_1^-, \dots, v_s^- \in A_1^-$, then

$$f(u_1^+, \dots, u_n^+, u_1^-, \dots, u_m^-, v_1^+, \dots, v_t^+, v_1^-, \dots, v_s^-) = 0.$$

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The $*$-algebra $F \oplus F$ generates a $*$-variety of almost polynomial growth.

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Let $\mathcal{V} = var^*(A)$ be a $*$-variety generated by a finite dimensional $*$-algebra over a field F of characteristic zero. Then $\mathcal{V}$ has polynomial growth if and only if $M, M^{sup}, F \oplus F \notin \mathcal{V}$.

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As a consequence:

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Let F be an algebraically closed field of characteristic zero and let $\mathcal{V}$ be a $*$-variety. Then $\mathcal{V}$ has polynomial growth if and only if $M, M^{sup}, F \oplus F, G^{\sharp}, G^{\star} \notin \mathcal{V}$.

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Definition

A $*$-variety $\mathcal{V}$ is said minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$, for some $k \geq 1$, $q > 0$, and for any proper subvariety, $\mathcal{U} \subseteq \mathcal{V}$, $c_n^*(\mathcal{U}) \approx q'n^t$, with $t < k$.

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- trivial grading
- superinvolution $*$ given by

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Theorem

C_k generates a minimal variety of polynomial growth.

Subvarieties of $\mathrm{var}^*(F \oplus F)$

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Theorem (**La Mattina, Martino 2016**, **I., La Mattina 2017**)

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3. $C \oplus N$,
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Subvarieties of var$^*(F \oplus F)$

Theorem (La Mattina, Martino 2016, I., La Mattina 2017)

Let A be a $*$-algebra such that $A \in var^*(F \oplus F)$. Then A is T_2^*-equivalent to one of the following algebras:

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3. $C \oplus N$,
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for some $k \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebras with trivial superinvolution.

The algebras A_k, N_k, U_k

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Let I the identity matrix and let $E = \sum_{i=2}^{k-1} e_{i,i+1} + e_{2k-i,2k-i+1}$.

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$$A_k = \text{span}\left\{e_{11} + e_{2k,2k}, E^j, e_{12}, \dots, e_{1k}, e_{k+1,2k}, \dots, e_{2k-1,2k}\right\}$$
$$N_k = \text{span}\left\{I, E^j, e_{12} - e_{2k-1,2k}, e_{13}, \dots, e_{1k}, e_{k+1,2k}, \dots, e_{2k-2,2k}\right\}$$
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Theorem

A_k, N_k, U_k generate minimal varieties of polynomial growth.

Subvarieties of $\mathrm{var}^*(M)$

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Theorem (La Mattina, Martino 2016, I., La Mattina 2017)

Let $A \in var^*(M)$ then A is T_2^*-equivalent to one of the following $*$-algebras:

Subvarieties of $\mathrm{var}^*(M)$

Theorem (**La Mattina, Martino 2016**, **I., La Mattina 2017**)

Let $A \in var^*(M)$ then A is T_2^*-equivalent to one of the following $*$-algebras:

$$M,\ N,\ N_k \oplus N,\ U_k \oplus N,\ N_k \oplus U_k \oplus N,$$

$$A_t \oplus N,\ N_k \oplus A_t \oplus N,\ U_k \oplus A_t \oplus N,\ N_k \oplus U_k \oplus A_t \oplus N,$$

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for some $k, t \geq 2$, where N is a nilpotent $*$-algebra.

The algebras A_k^{sup}, N_k^{sup}, U_k^{sup}

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We denote with A_k^{sup}, N_k^{sup} and U_k^{sup} the algebras A_k, N_k and U_k defined before, with

- elementary grading induced by

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- reflection superinvolution $\circ$.

Theorem (I., La Mattina 2017)

A_k^{sup}, N_k^{sup} and U_k^{sup} generate minimal varieties of polynomial growth.

Subvarieties of var$^*(M^{sup})$

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Theorem (I., La Mattina 2017)

If $A \in var^*(M^{sup})$ then A is T_2^*-equivalent to one of the following $*$-algebras:

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$$A_t^{sup} \oplus N,\ N_k^{sup} \oplus A_t^{sup} \oplus N,\ U_k^{sup} \oplus A_t^{sup} \oplus N,\ N_k^{sup} \oplus U_k^{sup} \oplus A_t^{sup} \oplus N,$$

Subvarieties of $\text{var}^*(M^{sup})$

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for some $k, t \geq 2$, where N is a nilpotent $*$-algebra and C is a commutative algebra with trivial superinvolution.

The algebras $G_k^\sharp$ and $G_k^\star$

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Theorem

$G_k^\sharp$ and $G_k^\star$ generate minimal varieties of polynomial growth.

Subvarieties of var$^*(G^\dagger)$

Subvarieties of $\mathrm{var}^*(G^\dagger)$

Let us denote by $\dagger$ one of the superinvolutions $\sharp$ and $\star$.

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Let A be a $*$-algebra such that $A \in \mathrm{var}^*(G^\dagger)$.

Subvarieties of $\mathrm{var}^*(G^\dagger)$

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Thank you