Growth of almost nilpotent varieties

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AGTA
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  2. A = nilpotent , $(A^n = 0, n \geq 1)$, then $x_1 \cdots x_n \equiv 0$ is a PI
  3. $A = M_2(F)$ then $[[x, y]^2, z] \equiv 0$ is a PI

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- $c_n(A) = \dim_F \frac{P_n}{P_n \cap Id(A)}$ is the n-th codimension of A.

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Examples

- Let $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ = the number of distinct arrangements of parentheses on a monomial of length n. Hence

$$c_n(F\{X\}) = \dim_F P_n = n! C_n = \binom{2n-2}{n-1}(n-1)!$$

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- If $F\langle X\rangle$ is the free associative algebra, then $c_n(F\langle X\rangle) = n!$
- For $L\langle X\rangle$ = the free Lie algebra, we have $c_n(L\langle X\rangle) = (n-1)!$.

Associative algebras

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- Kemer (1978). For an associative PI-algebra A, $c_n(A), n = 1, 2, \ldots,$ is either polynomially bounded , (i.e. $c_n(A) \leq \alpha n^t$,for some constants α, t), or grows exponentially.

$\mathcal{V} = var(A)$

$$\overline{\exp}(\mathcal{V}) = \operatorname{lím\,sup}_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}, \quad \underline{\exp}(\mathcal{V}) = \operatorname{lím\,ínf}_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}$$

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- Giambruno-Zaicev (1999). For an associative PI-algebra A,

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer.

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Definition. A variety $\mathcal{V}$ has intermediate growth if for any $k > 0$, $a > 1$ there exist constants C_1, C_2, such that for any n the inequalities

$$C_1 n^k < c_n(\mathcal{V}) < C_2 a^n$$

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hold.

Definition. A variety $\mathcal{V}$ has subexponential growth if for any constant B there exists n_0 such that for all $n > n_0$, $c_n(\mathcal{V}) < B^n$.

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Theorem (Mishchenho, Valenti)

Any non-nilpotent variety of algebras contains an almost nilpotent subvariety.

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2. $\mathcal{V}$ = a variety of Lie algebras.There is only one almost nilpotent variety: the metabelian variety, denoted $\mathcal{A}^2$ determined by the identity $[[x_1, x_2], [x_3, x_4]] \equiv 0$ In this case $c_n(\mathcal{A}^2) = n - 1$.

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For any integer $m \geq 2$, there exists an almost nilpotent variety $\mathcal{V}_m$ with $\exp(\mathcal{V}_m) = m$.

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We reach our objective in the setting of varieties satisfying the identity $x(yz) \equiv 0$.

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3. $\mathcal{V} = {}_2\mathcal{N}$.

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For any real number β, $0 < \beta < 1$, there exists a variety $\mathcal{V}_\beta \subseteq {}_2\mathcal{N}$, such that

$$\lim_{n \to \infty} \log_n \log_n c_n(\mathcal{V}_\beta) = \beta,$$

i.e. the sequence $c_n(\mathcal{V}_\beta)$ behaves like n^{n^β}, $n = 1, 2, \ldots$.

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Theorem (Mishchenko-Zaicev)

For any real number α, $3 < \alpha < 4$, there exists a variety of algebras $\mathcal{V}_\alpha \subseteq {}_2\mathcal{N}$ such that, for sufficiently large n, the following condition holds

$$C_1 n^\alpha < c_n(\mathcal{V}_\alpha) < C_2 n^\alpha,$$

where C_1, C_2 are positive constants.

Varieties with subexponential growth

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Let A_{alt} be the algebra over F generated by the countable set of elements $e_1, e_2, \ldots$ satisfying the following relations

1. $ue_ie_j = -ue_je_i$ for any nonempty word u in $e_1, e_2, \ldots$
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Remark

$\mathcal{V}_{alt} = \mathrm{var}(A_{alt}).$

Proposition

If $\mathcal{W} \subsetneqq \mathcal{V}_{sym}$ is a proper subvariety of $\mathcal{V}_{sym}$, then $\mathcal{W}$ is nilpotent.
If $\mathcal{W} \subsetneqq \mathcal{V}_{alt}$ is a proper subvariety of $\mathcal{V}_{alt}$, then $\mathcal{W}$ is nilpotent.

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Theorem (Mishckenko, Valenti)

Let $\mathcal{V}$ be a subvariety of ${}_2\mathcal{N}$. If $\mathcal{V}$ has subexponential growth then either $\mathcal{V}_{sym} \subseteq \mathcal{V}$ or $\mathcal{V}_{alt} \subseteq \mathcal{V}$ or $\mathcal{V}$ is nilpotent.

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Corollary

Let $\mathcal{V} \subset {}_2\mathcal{N}$ be an almost nilpotent variety. If $\mathcal{V}$ has subexponential growth then either $\mathcal{V} = \mathcal{V}_{sym}$ or $\mathcal{V} = \mathcal{V}_{alt}$.

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Theorem (Mishchenko-Valenti)

For any real number α, $0 < \alpha < 1$, the variety $\mathcal{V}_\alpha$ is an almost nilpotent variety and has linear or quadratic growth according as α is rational or irrational, respectively.

Theorem (Mishchenko-Valenti)

Over a field of characteristic zero there are countable many almost nilpotent metabelian varieties of at most linear growth and uncountable many almost nilpotent metabelian varieties of at most quadratic growth.

Thank You!

Definition

Given an infinite (associative) word w in the alphabet $\{0,1\}$ the complexity Comp_w of w is defined as the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of w of length n.

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Definition

w is called a Sturmian word if $\mathrm{Comp}_w(n) = n + 1$ for all $n \geq 1$. An infinite word $w = w_1 w_2 \cdots$ is periodic with period T if $w_i = w_{i+T}$ for $i = 1, 2, \ldots$.

Definition

Given an infinite (associative) word w in the alphabet $\{0,1\}$ the complexity Comp_w of w is defined as the function $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$, where $\mathrm{Comp}_w(n)$ is the number of distinct subwords of w of length n.

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Theorem (Mishchenko-Valenti)

For any real number α, $0 < \alpha < 1$, the variety $\mathcal{V}_\alpha$ has linear or quadratic growth according as w_α is a periodic or a Sturmian word.

Proposition

For $0 < \alpha < 1$, the variety $\mathcal{V}_\alpha$ is almost nilpotent.

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Theorem (Mishchenko-Valenti)

Over a field of characteristic zero there are countable many almost nilpotent metabelian varieties of at most linear growth and uncountable many almost nilpotent metabelian varieties of at most quadratic growth.

Thank You!!