On groups with finite conjugacy classes in a verbal subgroup

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To the organizers

Ancora Grazie, Thanks Again!

Let G be a group.

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Definition

A subgroup H of G is said to be FC-embedded in G if the set of conjugates

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- We are interested in the cases $H = w(G)$ or $w(G)'$, for a group word $w = w(x_1, \dots, x_n)$.

Recall that $w(G)$ is the verbal subgroup corresponding to w, that is, the subgroup generated by the set

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- If G is a $BFC(w)$-group, is $w(G)$ BFC-embedded in G?

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Let w be a concise word. Then G is an $FC(w)$-group if and only if $w(G)$ is FC-embedded in G.

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P. Hall - '60s

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- Is every n-Engel word concise?

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Similar questions arise when G is a $BFC(w)$-group.

Ivanov - 1984

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$$v(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n,$$

then $H_v = \{v_1 = 1, v_2\}$ and $Z(H) = \langle v_2 \rangle = v(H)$ is infinite.

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Let $G = H\, wr\, K$ where $K = \langle k \rangle$ is a cyclic group of order 2. Taking $w(x,y) = v(x^2, y^2)$, then $|x^{G_w}| \leq 4$ for all $x \in G$ and $k^{w(G)}$ is infinite. Thus G is a $BFC(w)$-group but $w(G)$ is not FC-embedded in G.

Taking G and w as before, one can check that $[w(G), G]$ is not FC-embedded in G.

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To all of you

Thanks for your attention!