

On groups with finite conjugacy classes in a verbal subgroup

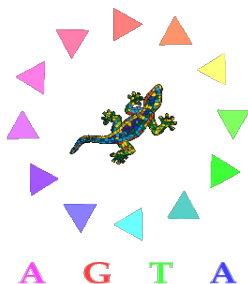
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(joint work with C. Delizia and P. Shumyatsky)

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Dipartimento di Matematica

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To the organizers



Ancora Grazie, Thanks Again!

Let G be a group.

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Definition

A subgroup H of G is said to be *FC*-embedded in G if the set of conjugates

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- We are interested in the cases $H = w(G)$ or $w(G)'$, for a group word $w = w(x_1, \dots, x_n)$.

Recall that $w(G)$ is the verbal subgroup corresponding to w , that is, the subgroup generated by the set

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P. Hall - '60s

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- Is every n -Engel word concise?

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Similar questions arise when G is a $BFC(w)$ -group.

Ivanov - 1984

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$$v(x, y) = [[x^{pn}, y^{pn}]^n, y^{pn}]^n,$$

then $H_v = \{v_1 = 1, v_2\}$ and $Z(H) = \langle v_2 \rangle = v(H)$ is infinite.

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Conjecture

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To all of you

Thanks for your attention!