Split reductive groups over rings and their relatives

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Let R be a ring (commutative, with 1) and I an ideal of R. Then there is a natural homomorphism $\rho_I : G(R) \to G(R/I)$, hence

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are normal subgroups.
Define $E(I) = \langle x_\alpha(r) \mid \alpha \in \Phi,\ r \in R \rangle$ and $\check{E}(R, I) = E(I)^{E(R)}$.
The latter is called the relative elementary subgroup.
If $G = SL_n$, then $x_\alpha(r)$ differs from the identity matrix in 1 nondiagonal place (α parametrizes the place, r is the entry).

Normal structure 2

Suppose that

- $\Phi \neq A_1$,
- 2 is invertible in R if $\Phi = B_n, C_n, F_4$, and
- 6 is invertible in R if $\Phi = G_2$.

Theorem

Given a subgroup $H \leqslant G(R)$, normalized by $E(R)$, there exists a unique ideal I of R such that

$$\check{E}(R, I) \leqslant H \leqslant \check{C}(R, I).$$

Moreover, $[\check{C}(R, I), E(R)] = \check{E}(R, I)$, hence all subgroups in the sandwich are normalized by $E(R)$.

Contents

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Results

1. Normality of $E(R)$ in $G(R)$ and commutator formulas.
2. Bounded width of commutators.
3. Nilpotent structure of $G(R)/E(R)$.

`http://alexei.stepanov.spb.ru/publicat.html`

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Axioms 1

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Property (generation)

$\check{E}(R, I)\check{E}(R, J) = \check{E}(R, I + J)$.
If $R = R' + I$, then $E(R')\check{E}(R, I) = E(R)$.

Axioms 2

Property (Gauss decomposition)

There exists an open cover of G by principal open subschemes $\mathcal{G}_i$, $i = 1, \ldots, m$, which are contained in E. (Note that $G(R) = \bigcup_{i=1}^{m} \mathcal{G}_i(R)$ for all *fields* R, not for all K-algebras).

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Property (clearing denominators)

Let $a \in \check{G}(R[t], tR[t])$. Suppose that $\lambda_S(a) \in E(R_S[t])$. Then there exists $s \in S$ such that $a(st) \in \check{E}(R[t], tR[t])$.

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Theorem (width of commutators)

There exists a constant $L \in \mathbb{N}$*, such that for any* K*-algebra* R*, ideal* I *of* R*,* $a \in G(R)$*, and* $b \in \check{E}(R,I)$ *(or* $a \in \check{G}(R,I)$ *and* $b \in \widetilde{E}(R)$ *) the commutator* $[a,b]$ *can be written as a product of* $\leqslant L$ *elements of* $\mathcal{S}(R,I)$*.*

Results 2

Let $I_1, \ldots, I_m$ be ideals of a K-algebra R.

Theorem (multicommutator formula)

$$[\check{E}(R, I_1), \check{G}(R, I_2), \ldots, \check{G}(R, I_m)] \leqslant$$
$$[\check{E}(R, I_1 \ldots I_{m-1}), \check{E}(R, I_m)] \cdot E(R, I_1 \ldots I_m) =: EE(R, I_1 \ldots I_{m-1}, I_m).$$

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Theorem (nilpotent structure of K_1)

If $\dim R \leqslant d$, *then*

$$[\check{G}(R, I_0), \check{G}(R, I_1), \ldots, \check{G}(R, I_d)] \leqslant EE(R, I_0 \ldots I_{d-1}, I_d).$$