Minimal varieties of graded PI algebras

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**Polynomial identities**

- $X := \{x_1, x_2, \ldots\}$ is a countable set and $F$ is a field
- $F\langle X \rangle$ is the free associative algebra over $F$ generated by $X$

**Definition**

An element $f(x_1, \ldots, x_s) \in F\langle X \rangle$ is a *polynomial identity* for an $F$-algebra $A$ if $f(a_1, \ldots, a_s) = 0_A$ for all $a_i \in A$.

If an $F$-algebra $A$ satisfies a non-trivial polynomial identity, then we say that $A$ is a *PI algebra*. 
Examples

- A *commutative* algebra is PI since it satisfies
  \[ [x_1, x_2] := x_1 x_2 - x_2 x_1 \]
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- A *nilpotent* algebra of degree \( n \) is PI since it satisfies
  \[ x_1 x_2 \cdots x_n \]
Examples

- A *commutative* algebra is PI since it satisfies
  \([x_1, x_2] := x_1 x_2 - x_2 x_1\)

- A *nilpotent* algebra of degree \(n\) is PI since it satisfies
  \(x_1 x_2 \cdots x_n\)

- A *finite dimensional* algebra of dimension \(n\) is PI since it satisfies
  \[\text{St}_{n+1}(x_1, x_2, \ldots, x_{n+1}) := \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma)x_{\sigma(1)} \cdots x_{\sigma(n+1)}\]
Examples

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  \[ [x_1, x_2] := x_1 x_2 - x_2 x_1 \]

- A **nilpotent** algebra of degree \( n \) is PI since it satisfies
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- A **finite dimensional** algebra of dimension \( n \) is PI since it satisfies
  \[
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  \]

- The **Grassmann algebra** on a countable dimension \( F \)-vector space (char \( F \neq 2 \)) is PI
A first line of research

Assume that $A$ is a PI algebra. What can one say on the algebraic structure of $A$?
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**Theorem [Isaacs-Passmann, 1973]**

Let $FG$ be the group algebra of a group $G$ over a field $F$ of characteristic $p$. Then $FG$ is PI if, and only if, $G$ has a $p$-abelian subgroup of finite index.
Focus on $\text{Id}(A)$

Describe

$$\text{Id}(A) := \{ f \mid f \in F\langle X \rangle \text{ f is PI for } A \}$$

for any PI algebra $A$
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The algebraic structure of $\text{Id}(A)$

$\text{Id}(A)$ is a $T$-ideal of $F\langle X \rangle$, namely a two-sided ideal closed under endomorphisms of $F\langle X \rangle$. 
The characteristic zero case and Specht’s Problem

Theorem [Kemer, 1991]
Let $F$ be a field of characteristic zero and $A$ a PI algebra. Then

$$\text{Id}(A) = \langle f_i \mid f_i \in F\langle X \rangle \text{ multilinear} \rangle_T.$$ 

Furthermore $\text{Id}(A)$ is finitely generated (Specht’s Problem).

So it is enough to consider

$$\bigoplus_{n \in \mathbb{N}} (P_n \cap \text{Id}(A)),$$

where $P_n := \text{span}_F \{x_{\tau(1)} \cdots x_{\tau(n)} \mid \tau \in S_n \}$. 


Polynomial identities for matrices

\[ \text{Id}(M_n(F)) = \langle [[[x_1, x_2]^2, x_3], \text{St}_4] \rangle_T \]
Polynomial identities for matrices

$$\text{Id}(M_2(F)) = \langle [[x_1, x_2]^2, x_3], \text{St}_4 \rangle_T$$

$$\text{Id}(M_n(F)) = \langle ? \rangle_T \text{ if } n \geq 3$$
Let us refine our thoughts: PI equivalent algebras

- In general, algebras satisfying the same polynomial identities are not isomorphic.
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**Definition**

Two $F$-algebras $A$ and $B$ are *PI-equivalent* if $\text{Id}(A) = \text{Id}(B)$.
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**Definition**

Given a $T$-ideal (set) $S$ of $F\langle X \rangle$, the class of all algebras $A$ such that $S \subseteq \text{Id}(A)$ for all $f \in S$ is called *the variety $\mathcal{V} = \mathcal{V}(S)$ determined by $S$*. Let us write $\text{Id}(\mathcal{V}) = S$. An algebra $A$ generates $\mathcal{V}$ if $\text{Id}(A) = \text{Id}(\mathcal{V})$ (write $\mathcal{V} = \text{var}(A)$).
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**Main Goal**

To classify varieties
Codimension Sequence

Definition [Regev, 1972]

Let $A$ be an $F$-algebra. The non-negative integer

$$c_n(A) := \dim_F \frac{P_n}{P_n \cap \Id(A)}$$

is said to be the $n$-th codimension of the algebra $A$.

- The sequence $(c_n(A))_{n \in \mathbb{N}}$ depends on $\Id(A)$ rather than $A$, thus it is constant on PI-equivalence classes and can therefore be used as an invariant.
The exponent of a PI algebra

**Theorem [Regev, 1972]**

If the algebra $A$ satisfies an identity of degree $d \geq 1$, then

$$c_n(A) \leq (d - 1)^{2n} \quad n \geq 1.$$
The exponent of a PI algebra

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Definition
Let $A$ be a PI algebra. Then the \textit{exponent} of $A$ is (if there exists)
\[ \exp(A) := \lim_{m \to +\infty} \sqrt[m]{c_m(A)}. \]
The exponent of a PI algebra

Theorem [Regev, 1972]

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Conjecture [Amitsur, ’80]

For any PI algebra $A$, $\exp(A)$ exists and is an integer.
Existence of the exponent

Let $A$ be a PI algebra. Then the exponent of $A$ exists and is an integer.

- They provide a method to compute it.
Importance of the exponent in classifying varieties

The most important feature of the exponent is that it provides an integral scale allowing to measure the growth of any variety.
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- Let $S \subseteq F\langle X \rangle$. It could be that if we consider a subset $S \subset T \subseteq F\langle X \rangle$ we get a strictly smaller variety with a strictly smaller exponent.
The most important feature of the exponent is that it provides an integral scale allowing to measure the growth of any variety.

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Let \( S \subseteq F\langle X \rangle \). It could be that if we consider a subset \( S \subseteq T \subseteq F\langle X \rangle \) we get a strictly smaller variety with a strictly smaller exponent.

If this always happens, \( \mathcal{V}(S) \) is called a minimal variety.
Minimal varieties

Definition
A variety $\mathcal{V}$ is *minimal* of exponent $d \geq 2$ if $\exp(\mathcal{V}) = d$ and $\exp(\mathcal{U}) < d$ for any proper subvariety $\mathcal{U} \subset \mathcal{V}$.

Theorem [Giambruno-Zaicev, 2003]
Let $\mathcal{V}$ be a variety of algebras of exponent $d \geq 2$. The following statements are equivalent:

(i) $\mathcal{V}$ is minimal;
(ii) $\text{Id}(\mathcal{V})$ is the product of verbally prime $T$-ideals;
(iii) $\mathcal{V} = \text{var}(\text{Gr}(A))$, where $A$ is a suitable *minimal* superalgebra and $\text{Gr}(A)$ is its Grassmann envelope.
Graded algebras

Definition

Let $G$ be a group and $F$ be a field. An (associative) $F$-algebra $A$ is called $G$-graded if

$$A = \bigoplus_{g \in G} A^{(g)},$$

where $A^{(g)}$ is an $F$-supaspace of $A$ and $A^{(g)}A^{(h)} \subseteq A^{(gh)}$ for every $g, h \in G$. When $G = \mathbb{Z}_2$, $A$ is said to be a superalgebra.
Examples

The Grassmann algebra

\[ Gr = \langle 1, e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \text{ for all } i, j \geq 1 \rangle \]

so a basis is given by

\[ B := \{1, e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k \}. \]

Set

\[ Gr^{(0)} := \text{span}\{e_{i_1} \cdots e_{i_{2k}} \mid 1 \leq i_1 < \ldots < i_{2k}, k \geq 0\}, \]

\[ Gr^{(1)} := \text{span}\{e_{i_1} \cdots e_{i_{2k+1}} \mid 1 \leq i_1 < \ldots < i_{2k+1}, k \geq 0\}. \]

Then \( Gr = Gr^{(0)} \oplus Gr^{(1)} \). In particular, \( Gr^{(0)} = Z(Gr) \).
Let $G$ be a group and $(g_1, \ldots, g_n) \in G^n$. Consider the algebra $M_n(F)$ graded by

$$
\begin{pmatrix}
g_1^{-1}g_1 = 1_G & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\
g_2^{-1}g_1 & g_2^{-1}g_2 = 1_G & g_2^{-1}g_3 & \cdots & g_2^{-1}g_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & \cdots & g_n^{-1}g_n = 1_G
\end{pmatrix}.
$$

This grading on $M_n(F)$ is called *elementary*. 
Graded polynomial identities

- $G$ is a group
- $F\langle X \rangle^G$ is the free associative $G$-graded algebra of countable rank over $F$ (here, if $G = \{g_1, g_2, \ldots\}$, the set $X$ decomposes as $X = \bigcup_{i=1}^{s} X^{(g_i)}$, where $X^{(g_i)} = \{x_1^{(g_i)}, x_2^{(g_i)}, \ldots\}$)

**Definition**

An element $f(x_1^{(g_1)}, \ldots, x_{t_1}^{(g_1)}, \ldots, x_1^{(g_s)}, \ldots, x_{t_s}^{(g_s)}) \in F\langle X \rangle^{gr}$ is a **graded polynomial identity** for the $G$-graded algebra $A$ if $f(a_1^{(g_1)}, \ldots, a_{t_1}^{(g_1)}, \ldots, a_1^{(g_s)}, \ldots, a_{t_s}^{(g_s)}) = 0_A$ for all $a_i \in A$. 
Let us consider the $T_G$-ideal of graded polynomial identities of a $G$-graded algebra $A$

$$\text{Id}_G(A) := \{ f | f \in F\langle X \rangle^G \text{ multilinear} \}.$$ 

**Theorem [Aljadeff-Kanel Belov, 2010]**

Let $F$ be a field of **characteristic zero** and $A$ be a PI algebra graded by a **finite** group $G$. Then

$$\text{Id}_G(A) = \langle f_i | f_i \in F\langle X \rangle^{gr} \text{ multilinear} \rangle_{T_G}.$$ 

Furthermore $\text{Id}_G(A)$ is finitely generated (**Graded Specht’s Problem**).
Assume that char $F = 0$. As in the ungraded case, it is enough to consider the spaces of multilinear $G$-graded polynomials in the variables $x_1^{(g_{i_1})}, \ldots, x_n^{(g_{i_n})}$

$$P_n^G := \text{Span}\{x_1^{(g_{i_1})} \cdots x_n^{(g_{i_n})} | \sigma \in S_n \quad g_{i_1}, \ldots, g_{i_n} \in G\}.$$ 

**Definition**

The non-negative integer

$$c_n^G(A) := \dim_F \frac{P_n^G}{P_n^G \cap \text{Id}_G(A)}$$

is said to be the *n-th* $G$-graded codimension of the algebra $A$. 

Polynomial identities versus graded polynomial identities

- If $A$ is a $G$-graded algebra which is PI, then it satisfies a graded polynomial identity.
Polynomial identities versus graded polynomial identities

- If $A$ is a $G$-graded algebra which is PI, then it satisfies a graded polynomial identity.
- The converse is, in general, not true: it is enough to consider the free algebra generated by two indeterminates with the trivial grading.
Polynomial identities versus graded polynomial identities

- If $A$ is a $G$-graded algebra which is PI, then it satisfies a graded polynomial identity.
- The converse is, in general, not true: it is enough to consider the free algebra generated by two indeterminates with the trivial grading.

**Theorem [Giambruno-Regev, 1985]**

If $A$ is a PI-algebra graded by a finite group $G$, then

$$c_n(A) \leq c_n^G(A) \leq |G|^n c_n(A) \quad n \geq 1.$$
The graded exponent

**Definition**

Let $A$ be a PI algebra graded by a finite group $G$. Then *the graded exponent* of $A$ is (if there exists)

$$
\exp^G(A) := \lim_{m \to +\infty} \sqrt[m]{c_m^G(A)}.
$$

Benanti-Giambruno-Pipitone [J. Algebra 269 (2003), 422–438]: $G = \mathbb{Z}_2$ and $A$ is finitely generated

Aljadeff-Giambruno-La Mattina [J. Reine Angew. Math. 650 (2011), 83–100]: $G$ is finite abelian and $A$ is finite-dimensional

Giambruno-La Mattina [Adv. Math. 225 (2010), 859–881]: $G$ is finite abelian and $A$ is PI
The graded exponent

Definition

Let $A$ be a PI algebra graded by a finite group $G$. Then the \textit{graded exponent} of $A$ is (if there exists)

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Theorem [Aljadeff-Giambruno, 2013]

Let $A$ be a PI algebra graded by a finite group $G$. Then the graded exponent of $A$ exists and is an integer.
The graded problem

Classify the minimal varieties of PI algebras graded by a finite group $G$ of fixed graded exponent.

Definition

A variety $\mathcal{V}^G$ of $G$-graded PI algebras is said to be minimal of graded exponent $d$ if $\exp^G(\mathcal{V}^G) = d$ and $\exp^G(\mathcal{U}^G) < d$ for every proper subvariety $\mathcal{U}^G \subset \mathcal{V}^G$. 
Motivations

- To construct a theory which generalizes that of ordinary polynomial identities.
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- To construct a theory which generalizes that of ordinary polynomial identities.
- The additional graded structure and related objects may provide significant information on quite general objects. For instance, Kemer in his fundamental work [Transl. Math. Monograph, vol. 87, Amer. Math. Soc., Providence, RI, 1991] proved that for any non-trivial variety there exists a finite-dimensional superalgebras $A$ (on an extension of $F$) such that the T-ideal of the variety coincides with the T-ideal of polynomial identities of $Gr(A)$. 
Theorem [Di Vincenzo-S., 2012]

Let $\mathcal{V}_{\mathbb{Z}_2}$ be a variety PI superalgebras of finite basic rank. If $\mathcal{V}_{\mathbb{Z}_2}$ is minimal of $\mathbb{Z}_2$-graded exponent $d \geq 2$, then $\mathcal{V}_{\mathbb{Z}_2} = \text{var}_{\mathbb{Z}_2}(B)$, where $B$ is a suitable minimal superalgebra.
Definition [Giambruno-Zaicev, 2003]

Let $F$ be an algebraically closed field. An $F$-superalgebra $A$ is called **minimal** if it is finite dimensional and $A = A_{ss} + J$ where

1. $A_{ss} = A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n$ simple superalgebras;
2. there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J^{(0)} \cup J^{(1)}$ and minimal graded idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that
   
   \[
   e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}, \quad i = 1, \ldots, n - 1
   \]

   and

   \[
   w_{12} w_{23} \cdots w_{n-1,n} \neq 0;
   \]
3. $w_{12}, \ldots, w_{n-1,n}$ generate $J$ as two-sided ideal of $A$.  

Simple Superalgebras

- \( M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), where \( k \geq l \geq 0, k \neq 0, A \in M_k, D \in M_l, B \in M_{k \times l} \) and \( C \in M_{l \times k} \), endowed with the grading \( M_{k,l}^{(0)} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \) and \( M_{k,l}^{(1)} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \);

- \( M_m(F \oplus tF) \), where \( t^2 = 1 \) with grading \( (M_m, tM_m) \),

where, for any pair of positive integers \( m \) and \( s \), the symbol \( M_{m \times s} \) means the \( F \)-vector space of all rectangular matrices with \( m \) rows and \( s \) columns, and \( M_m := M_{m \times m} \).
Simple graded simple components

Proposition

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$, where $A_j = M_{k_j,l_j}$ for all $j$, then $A$ is isomorphic (as superalgebra) to $UT(k_1 + l_1, \ldots, k_n + l_n)$ equipped with a suitable elementary grading.
Simple graded simple components

**Proposition**

Let \( A = A_{ss} + J \) be a minimal superalgebra. If \( A_{ss} = A_1 \oplus \cdots \oplus A_n \), where \( A_j = M_{k_j,l_j} \) for all \( j \), then \( A \) is isomorphic (as superalgebra) to \( UT(k_1 + l_1, \ldots, k_n + l_n) \) equipped with a suitable elementary grading.

**Open question**

Is it true that the supervariety generated by a minimal superalgebra is minimal with respect to its graded exponent?
Strategy

- Let $A$ be a minimal superalgebra and let $\mathcal{V}_{\mathbb{Z}_2} := \text{var}_{\mathbb{Z}_2}(A)$. 
Let $A$ be a minimal superalgebra and let $\mathcal{V}_{\mathbb{Z}_2} := \text{var}_{\mathbb{Z}_2}(A)$. Let $\mathcal{U}_{\mathbb{Z}_2} \subseteq \mathcal{V}_{\mathbb{Z}_2}$ such that $\exp_{\mathbb{Z}_2}(\mathcal{V}_{\mathbb{Z}_2}) = \exp_{\mathbb{Z}_2}(\mathcal{U}_{\mathbb{Z}_2})$. 

We aim to show that $\mathcal{U}_{\mathbb{Z}_2} = \mathcal{V}_{\mathbb{Z}_2}$. Now $\mathcal{U}_{\mathbb{Z}_2}$ has finite basic rank. Hence, by Kemer’s result, $\mathcal{U}_{\mathbb{Z}_2}$ is generated by a finite dimensional superalgebra $B'$. On the other hand, there exists a minimal superalgebra $B$ such that $\text{Id}_{\mathbb{Z}_2}(B') \subseteq \text{Id}_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(B') = \exp_{\mathbb{Z}_2}(B)$. Consequently, $\text{Id}_{\mathbb{Z}_2}(A) \subseteq \text{Id}_{\mathbb{Z}_2}(B)$ and $\exp_{\mathbb{Z}_2}(A) = \exp_{\mathbb{Z}_2}(B)$. This implies that $A_{ss} = B_{ss}$. We have to prove that $\text{Id}_{\mathbb{Z}_2}(A) = \text{Id}_{\mathbb{Z}_2}(B)$.
Let $A$ be a minimal superalgebra and let $\mathcal{V}_{\mathbb{Z}_2} := \text{var}_{\mathbb{Z}_2}(A)$.

Let $\mathcal{U}_{\mathbb{Z}_2} \subseteq \mathcal{V}_{\mathbb{Z}_2}$ such that $\exp_{\mathbb{Z}_2}(\mathcal{V}_{\mathbb{Z}_2}) = \exp_{\mathbb{Z}_2}(\mathcal{U}_{\mathbb{Z}_2})$.

We aim to show that $\mathcal{U}_{\mathbb{Z}_2} = \mathcal{V}_{\mathbb{Z}_2}$. 
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We aim to show that $\mathcal{U}^{\mathbb{Z}_2} = \mathcal{V}^{\mathbb{Z}_2}$.

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Let $\mathcal{U}_{\mathbb{Z}^2} \subseteq \mathcal{V}_{\mathbb{Z}^2}$ such that $\exp_{\mathbb{Z}^2}(\mathcal{V}_{\mathbb{Z}^2}) = \exp_{\mathbb{Z}^2}(\mathcal{U}_{\mathbb{Z}^2})$.

We aim to show that $\mathcal{U}_{\mathbb{Z}^2} = \mathcal{V}_{\mathbb{Z}^2}$.

Now $\mathcal{U}_{\mathbb{Z}^2}$ has finite basic rank. Hence, by Kemer’s result, $\mathcal{U}_{\mathbb{Z}^2}$ is generated by a finite dimensional superalgebra $B'$.

On the other hand, there exists a minimal superalgebra $B$ such that $\text{Id}_{\mathbb{Z}^2}(B') \subseteq \text{Id}_{\mathbb{Z}^2}(B)$ and $\exp_{\mathbb{Z}^2}(B') = \exp_{\mathbb{Z}^2}(B)$.
Let $A$ be a minimal superalgebra and let $V\mathbb{Z}_2 := \text{var}^\mathbb{Z}_2(A)$.

Let $U\mathbb{Z}_2 \subseteq V\mathbb{Z}_2$ such that $\exp^\mathbb{Z}_2(V\mathbb{Z}_2) = \exp^\mathbb{Z}_2(U\mathbb{Z}_2)$.

We aim to show that $U\mathbb{Z}_2 = V\mathbb{Z}_2$.

Now $U\mathbb{Z}_2$ has finite basic rank. Hence, by Kemer’s result, $U\mathbb{Z}_2$ is generated by a finite dimensional superalgebra $B'$.

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Consequently, $\text{Id}_{\mathbb{Z}_2}(A) \subseteq \text{Id}_{\mathbb{Z}_2}(B)$ and $\exp^\mathbb{Z}_2(A) = \exp^\mathbb{Z}_2(B)$. 

This implies that $A = B'$.
Strategy

- Let $A$ be a minimal superalgebra and let $\mathcal{V}_{\mathbb{Z}^2} := \text{var}_{\mathbb{Z}^2}(A)$.
- Let $\mathcal{U}_{\mathbb{Z}^2} \subseteq \mathcal{V}_{\mathbb{Z}^2}$ such that $\exp_{\mathbb{Z}^2}(\mathcal{V}_{\mathbb{Z}^2}) = \exp_{\mathbb{Z}^2}(\mathcal{U}_{\mathbb{Z}^2})$.
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- On the other hand, there exists a minimal superalgebra $B$ such that $\text{Id}_{\mathbb{Z}^2}(B') \subseteq \text{Id}_{\mathbb{Z}^2}(B)$ and $\exp_{\mathbb{Z}^2}(B') = \exp_{\mathbb{Z}^2}(B)$.
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- This implies that $A_{ss} = B_{ss}$. 
Strategy

- Let $A$ be a minimal superalgebra and let $\mathcal{V}_{\mathbb{Z}_2} := \text{var}_{\mathbb{Z}_2}(A)$.
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- This implies that $A_{ss} = B_{ss}$.
- We have to prove that $\text{Id}_{\mathbb{Z}_2}(A) = \text{Id}_{\mathbb{Z}_2}(B)$. 
Proposition [Di Vincenzo-S., 2012]

Let $A := (UT(\alpha_1, \ldots, \alpha_n), | |_A)$ and $B := (UT(\alpha_1, \ldots, \alpha_n), | |_B)$. If $\text{Id}_{\mathbb{Z}_2}(A) \subseteq \text{Id}_{\mathbb{Z}_2}(B)$, then $A \cong B$. 
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- This result has been generalized in Di Vincenzo-Spinelli [J. Algebra 415 (2014), 50–64] for gradings on upper block triangular matrix algebras induced by a finite abelian group under suitable restrictions.
- David [J. Algebra 367 (2012), 120–141]: semisimple $G$-graded algebras
Theorem [Di Vincenzo-S., 2012]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$, where $A_j = M_{k_j,l_j}$ for all $j$, then $\text{var}_{\mathbb{Z}_2}(A)$ is minimal of graded exponent $\dim_F A_{ss}$.
Giambruno-Zaicev [Adv. Math. 174 (2003), 310–323] proved that a variety of finite basic rank is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(d_1, \ldots, d_n)$ and

$$\text{Id}(UT(d_1, \ldots, d_n)) = \text{Id}(M_{d_1}) \cdots \text{Id}(M_{d_n})$$
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Minimal superalgebras in which all the summands of the semisimple part are simple graded simple generate minimal supervarieties, but in general they do not generate the same supervariety not even if they have the same graded components $A_1, \ldots, A_n$. Hence we cannot hope that the $T_2$-ideal of superidentities of these minimal superalgebras is factorable.
The case with two graded simple summands

Theorem [Di Vincenzo-S., 2012]

Let \( A = A_{ss} + J \) be a minimal superalgebra such that \( A_{ss} = A_1 \oplus A_2 \). Then \( \text{Id}_{\mathbb{Z}_2}(A) = \text{Id}_{\mathbb{Z}_2}(A_1) \cdot \text{Id}_{\mathbb{Z}_2}(A_2) \) if one of the following conditions is satisfied:

- at least one between \( A_1 \) and \( A_2 \) is non-simple as algebra;
- \( A_1 \) and \( A_2 \) are both simple \( \mathbb{Z}_2 \)-simple and there exists \( 1 \leq i \leq 2 \) such that \( A_i = M_{k_i,k_i} \).
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Theorem [Di Vincenzo-S., 2012]
Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$. Then the supervariety generated by $A$ is minimal of graded exponent $\dim_F(A_1 \oplus A_2)$. 
On the structure of minimal superalgebras

- It has been shown that a minimal superalgebra $A = (A_1 \oplus \ldots \oplus A_n) + J$ has the following vector space decomposition:

$$A = \bigoplus_{1 \leq i \leq j \leq n} A_{ij},$$

where $A_{11} := A_1, \ldots, A_{nn} := A_n$ and, for all $i < j$,

$$A_{ij} := A_i w_{i, i+1} A_{i+1} \cdots A_{j-1} w_{j-1, j} A_j.$$

Moreover $J = \bigoplus_{i < j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where $\delta_{jk}$ is the Kronecker delta.

- For every $1 \leq k < l \leq n$ set

$$A^{(k,l)} := \bigoplus_{k \leq i \leq j \leq l} A_{ij},$$

which is a minimal superalgebra as well.
Theorem [Di Vincenzo-da Silva-S., 2016]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \ldots, A_h$ are non-simple graded simple and $A_{h+1}, \ldots, A_n$ are simple graded simple algebras, then

$$\id_{\mathbb{Z}_2}(A) = \id_{\mathbb{Z}_2}(A_1) \cdots \id_{\mathbb{Z}_2}(A_h) \cdot \id_{\mathbb{Z}_2}(A^{(h+1,n)}).$$

On the other hand, if $h < n$ and $A_1, \ldots, A_h$ are simple graded simple and $A_{h+1}, \ldots, A_n$ are non-simple graded simple algebras, then

$$\id_{\mathbb{Z}_2}(A) = \id_{\mathbb{Z}_2}(A^{(1,h)}) \cdot \id_{\mathbb{Z}_2}(A_{h+1}) \cdots \id_{\mathbb{Z}_2}(A_n).$$
Another positive result

Theorem [Di Vincenzo-da Silva-S., 2016]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \ldots, A_h$ are non-simple graded simple and $A_{h+1}, \ldots, A_n$ are simple graded simple algebras (or vice versa), then the supervariety generated by $A$ is minimal of graded exponent $\text{dim}_F(A_{ss})$. 
A crucial example

Consider $R := UT_6$ endowed with the $\mathbb{Z}_2$-grading induced by the automorphism $\phi$ (of order 2) defined on $E_{ij}$ by

$$\phi(E_{ij}) := E_{\rho(i),\rho(j)}, \quad \rho := (12)(34)(56).$$

Take the subalgebra $A$ of $R$ having as a linear basis the set

$$\mathcal{B}_A := \{E_{11} + E_{22}, E_{33}, E_{44}, E_{55} + E_{66}, E_{13}, E_{24}, E_{35}, E_{46}, E_{15}, E_{26}\},$$

which is homogeneous. Its Wedderburn-Malcev decomposition is $A_{ss} = A_1 \oplus A_2 \oplus A_3$, where

- $A_1 = \langle E_{11} + E_{22} \rangle \cong F (e_1 := E_{11} + E_{22})$;
- $A_2 = \langle E_{33}, E_{44} \rangle \cong F \oplus tF$, where $t^2 = 1$, with grading $(F, tF) (e_2 := E_{33} + E_{44})$;
- $A_3 = \langle E_{55} + E_{66} \rangle \cong F (e_3 := E_{55} + E_{66})$.

and Jacobson radical $J(A) = \langle E_{13}, E_{24}, E_{35}, E_{46}, E_{15}, E_{26} \rangle$ is generated as an ideal by the homogeneous elements

$$w_{12} := E_{13} + E_{24}, \quad w_{23} := E_{35} + E_{46}.$$
Consider $S := UT_4$ endowed with the $\mathbb{Z}_2$-grading induced by the automorphism $\psi$ defined by

$$\psi(E_{ij}) := E_{\sigma(i),\sigma(j)}, \quad \sigma := (23).$$

Take the subalgebra $B$ of $S$ having as a linear basis the set

$$B_B := \{ E_{11}, E_{22}, E_{33}, E_{44}, E_{12}, E_{13}, E_{24}, E_{34}, E_{14} \}.$$

$B$ is a minimal superalgebra with graded simple summands of $B_{ss}$, where

- $B_1 = \langle E_{11} \rangle \cong F$ ($e_1 := E_{11}$);
- $B_2 = \langle E_{22}, E_{33} \rangle \cong F \oplus tF$, where $t^2 = 1$, with grading $(F, tF)$ ($e_2 := E_{22} + E_{33}$);
- $B_3 = \langle E_{44} \rangle \cong F$ ($e_3 := E_{44}$)

and Jacobson radical $J(B) = \langle E_{12}, E_{13}, E_{24}, E_{34}, E_{14} \rangle$ generated as an ideal by the homogeneous elements

$$w_{12} := E_{12} + E_{13}, \quad w_{23} := E_{24} + E_{34}.$$
$A_{ss} = B_{ss}$ and $\exp^{\mathbb{Z}_2}(A) = 4 = \exp^{\mathbb{Z}_2}(B)$
\[ A_{ss} = B_{ss} \text{ and } \exp_{\mathbb{Z}_2}(A) = 4 = \exp_{\mathbb{Z}_2}(B) \]

There is a graded epimorphism from \( A \) to \( B \). Consequently,

\[ \text{Id}_{\mathbb{Z}_2}(A) \subseteq \text{Id}_{\mathbb{Z}_2}(B) \]
\begin{itemize}
  \item $A_{ss} = B_{ss}$ and $\exp_{\mathbb{Z}_2}(A) = 4 = \exp_{\mathbb{Z}_2}(B)$
  \item There is a graded epimorphism from $A$ to $B$. Consequently,
    \[ \text{Id}_{\mathbb{Z}_2}(A) \subseteq \text{Id}_{\mathbb{Z}_2}(B) \]
  \item The graded polynomial $g := [y_1, y_2]z_3[y_4, y_5]$ lies in $\text{Id}_{\mathbb{Z}_2}(B)$ but $[e_1, w_{12}](E_{33} - E_{44})[w_{23}, e_3] \neq 0_A$. Hence
    \[ \text{Id}_{\mathbb{Z}_2}(A) \neq \text{Id}_{\mathbb{Z}_2}(B) \]
\end{itemize}
- $A_{ss} = B_{ss}$ and $\exp^{\mathbb{Z}_2}(A) = 4 = \exp^{\mathbb{Z}_2}(B)$
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- The graded polynomial $g := [y_1, y_2]z_3[y_4, y_5]$ lies in $\text{Id}_{\mathbb{Z}_2}(B)$ but $[e_1, w_{12}](E_{33} - E_{44})[w_{23}, e_3] \neq 0_A$. Hence

\[ \text{Id}_{\mathbb{Z}_2}(A) \neq \text{Id}_{\mathbb{Z}_2}(B) \]

- $\text{var}^{\mathbb{Z}_2}(A)$ is not minimal.
The case with three graded simple addends

Di Vincenzo-da Silva-Spinelli [Math. Z., in press] have completely described the isomorphism types of minimal superalgebras whose maximal semisimple homogeneous subalgebra is the sum of three graded simple algebras.
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A minimal superalgebra $A$ belonging to this class generates a minimal supervariety of fixed graded exponent except for one case.
Theorem [Di Vincenzo-da Silva-S., 2017]

Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ with

$$A_1 = M_{k,l}, \quad A_2 = M_m(F \oplus tF) \quad \text{and} \quad A_3 = M_{r,s}.$$

(a) If $A_{13}$ is irreducible as an $(A_1, A_3)$-bimodule, then $A$ generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$;

(b) if $A_{13}$ is not irreducible as an $(A_1, A_3)$-bimodule, then $A$ generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$ if, and only if, either $k = l$ or $r = s$. 
Definition

Let $F$ be an algebraically closed field. A $G$-graded algebra $A$ is called *minimal* if it is finite-dimensional and $A = A_{ss} + J(A)$ where

(i) $A_{ss} = A_1 \oplus \cdots \oplus A_n$, with $A_1, \ldots, A_n$ $G$-simple algebras;

(ii) there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J(A)$ and minimal homogeneous idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that

$$e_i w_{i,i+1} = e_{i+1} w_{i,i+1} = w_{i,i+1} \quad 1 \leq i \leq n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;$$

(iii) $w_{12}, \ldots, w_{n-1,n}$ generate $J(A)$ as a two-sided ideal of $A$. 
\( \mathbb{Z}_p \)-simple algebras

Let \( G = \langle \epsilon \rangle \cong \mathbb{Z}_p \) and

\[
D = \begin{pmatrix}
\epsilon & a_2 & \cdots & a_{p-1} & a_p \\
\epsilon & a_1 & \cdots & a_{p-1} & a_p \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_3 & \cdots & \cdots & a_2 & \epsilon \\
a_2 & a_3 & \cdots & a_p & a_1
\end{pmatrix}
\]

, where \( a_1, a_2, \ldots, a_p \in F \),

with its natural grading induced by the \( p \)-tuple \( (1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}) \).
Proposition

Let $F$ be an algebraically closed field and $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ a group of prime order $p$. If $A$ is a finite-dimensional $G$-simple algebra, then it is isomorphic to one of the following $G$-graded algebras:

(i) $M_n$ with an elementary grading;

(ii) $D \otimes M_r$ with the grading induced by the trivial grading on $M_r$ and the natural one on $D$. In other words, in this case, $A$ is isomorphic to the homogeneous subalgebra $M_r(D)$ of $M_{pr}$ with the grading induced by the $(pr)$-tuple $(1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}, \ldots, 1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1})$. 

$r$ times
The graded algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$

- Assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and $F$ is algebraically closed.
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Given an $m$-tuple $(A_1, \ldots, A_m)$ of $G$-simple algebras, let

$$\Gamma_0 := \{ i \mid A_i \text{ is simple graded simple} \}, \quad \Gamma_1 := [1, m] \setminus \Gamma_0.$$
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  \[
  \Gamma_0 := \{ i \mid A_i \text{ is simple graded simple} \}, \quad \Gamma_1 := [1, m] \setminus \Gamma_0.
  \]

- For every $k \in [1, m]$, let us denote the size of $A_k$ by
  \[
  s_k := \begin{cases} 
  n_k & \text{if } k \in \Gamma_0 \text{ and } A_k \cong M_{n_k}, \\
  p_{n_k} & \text{if } k \in \Gamma_1 \text{ and } A_k \cong M_{n_k}(D) \subseteq M_{p_{n_k}}
  \end{cases}
  \]
  and set $\nu_0 := 0$, $\nu_k := \sum_{i=1}^{k} s_i$ and $\text{Bl}_k := [\nu_{k-1} + 1, \nu_k]$
Let

\[ UT(A_1, \ldots, A_m) := \{(a_{ij}) \in UT(s_1, \ldots, s_m) | a_{kk} \in M_{n_k}(D), k \in \Gamma_1\} \]

and \( \alpha_k : [1, s_k] \rightarrow G \) be the map inducing the elementary grading on \( A_k \). In particular, if \( k \in \Gamma_1 \),

\[ (\alpha_k(1), \ldots, \alpha_k(s_k)) := (1_G, \epsilon, \ldots, \epsilon^{p-1}, \ldots, 1_G, \epsilon, \ldots, \epsilon^{p-1}) \]

\( n_k \) times.
Let

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and \( \alpha_k : [1, s_k] \longrightarrow G \) be the map inducing the elementary grading on \( A_k \). In particular, if \( k \in \Gamma_1 \),

\[
(\alpha_k(1), \ldots, \alpha_k(s_k)) := (1^G, \epsilon, \ldots, \epsilon^{p-1}, \ldots, 1^G, \epsilon, \ldots, \epsilon^{p-1})^\text{n\_k times}.
\]

Let us define the maps

\[
\alpha : [1, \nu_m] \longrightarrow G, \quad i \longmapsto \alpha_k(i - \nu_{k-1})
\]

and, for any \( m \)-tuple \( \tilde{g} := (g_1, \ldots, g_m) \in G^m \),

\[
\alpha_{\tilde{g}} : [1, \nu_m] \longrightarrow G, \quad i \longmapsto g_k \alpha(i),
\]

where \( k \in [1, m] \) is the (unique) integer such that \( i \in \text{Bl}_k \).
Let

\[ UT(A_1, \ldots, A_m) := \{(a_{ij}) \in UT(s_1, \ldots, s_m) \mid a_{kk} \in M_{n_k}(D), k \in \Gamma_1\} \]

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Let us denote any such \( G \)-graded algebra by \( UT_G(A_1, \ldots, A_m) \).
Theorem [Di Vincenzo-da Silva-S., 2017]

Let $F$ be a field of characteristic zero and $G$ a group of prime order $p$. A variety of $G$-graded PI-algebras of finite basic rank is minimal of $G$-exponent $d$ if, and only if, it is generated by a $G$-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$. 

