Hilbert series of noncommutative structures, regular languages and symmetric functions

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AGTA (Lecce, 8.9.2017)

Based on publications:

—— , Monomial right ideals and the Hilbert series of noncommutative modules. J. Symbolic Comput. 80 (2017)

—— , Tiwari, Sharwan K., Multigraded Hilbert Series of noncommutative modules.

arXiv:1705.01083 (2017)

- The Hilbert series of algebras and modules are an extremely useful tool, both in the graded (univariate) and multigraded (multivariate) case.
- When such series have rational sums, we have polynomial or exponential growths and we can compute by them polynomial and exponential dimensions.
- They also provide other kinds of growth when they are irrational.
- The Hilbert series are used to confirm the homological structure.
- When an algebra is invariant under the action of the general linear group, the sum of its multigraded Hilbert series is a symmetric function providing the module structure via the Schur function decomposition.
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- To illustrate some of these applications together with the proposed methods for computing Hilbert series, we start with two examples for the graded and multigraded case.
- The general approach here is to reduce the computation to the monomial case.
- If $A = F/I$ is a finitely generated associative algebra (F the free associative algebra and I the two-sided ideal of the relations satisfied by the generators of A) then the Hilbert series of A coincides with the corresponding series of $A' = F/I'$ where I' is the leading monomial ideal of I (with respect to a monomial ordering of F).
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Example

- Consider an Artin group defined by the following Coxeter matrix

$$C = \begin{pmatrix} - & \infty & 3 \\ \infty & - & 2 \\ 3 & 2 & - \end{pmatrix}$$

that is, the group has the following presentation

$$G = \langle x, y, z \mid yz = zy, xzx = zxz \rangle.$$

- We consider now the algebra A which has the same presentation. In other words, $A = F/J$ where $F = \mathbb{K}\langle x, y, z \rangle$ and $J \subset F$ is the following two-sided ideal

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- We compute (formal arguments) an *infinite* Gröbner basis $\mathcal{G}$ of J with respect to the graded lexicographic monomial ordering of F

$$\mathcal{G} = \{yz - zy, xzx - zxz\} \cup \{xz^2z^dxz - zxz^2xx^d \mid d \geq 0\}.$$

- This implies that the leading monomial ideal I of J is the following *infinitely generated* ideal

$$I = \langle yz, xzx \rangle + \langle xz^2z^dxz \mid d \geq 0 \rangle.$$

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$$\{1\},\{x,y,z\},\{x^2,xy,xz,yx,y^2,zx,zy,z^2\},$$
$$\{x^3,x^2y,x^2z,xyx,xy^2,xzy,xz^2,yx^2,yxy,yxz,y^2x,y^3,$$
$$zx^2,zxy,zxz,zyx,zy^2,z^2x,z^2y,z^3\},\ldots$$

- Then, the *Hilbert function* $d \mapsto \dim C_d$ is

$$1,3,8,20,49,119,288,696,1681,4059,9800,23660,57121,$$
$$137903,332928,803760,\ldots$$

- By our methods, we compute (formal arguments) that the corresponding generating function, the *Hilbert series* of C is

$$\mathrm{HS}(C)=\sum_{d\geq 0}\dim C_d t^d=\frac{1}{(1-t)(1-2t-t^2)}$$

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Example

- Note that one may use also the notion of *growth* or *affine Hilbert function* (which is necessary when the algebra is a non-graded one)

$$d \mapsto \dim A_{\leq d} = \dim C_{\leq d} = \sum_{k \leq d} \dim C_k.$$

- The corresponding generating function, the *affine Hilbert series* is immediately related to the previous series as

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- In particular, for the considered (graded) algebra A we have the affine Hilbert series

$$\mathrm{HS}_a(A) = \sum_{d \geq 0} \dim A_{\leq d} t^d = \frac{1}{(1-t)^2(1-2t-t^2)}$$

- Since the roots of the denominator are $1, -1 \pm \sqrt{2}$, note finally that the algebra A has an exponential growth with exponential dimension

$$\limsup_{d \to \infty} \sqrt[d]{\dim A_{\leq d}} = \frac{1}{-1+\sqrt{2}}$$

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- This new example shows that the computation of (multigraded) Hilbert series can be applied for the purposes of representation theory, algebraic combinatorics, etc. For this example we make explicit our method.
- Let $E = \bigwedge(V)$ be the Grassmann (or exterior) algebra over a vector space V of countable dimension. If $F = \mathbb{K}\langle x_1, \ldots, x_n\rangle$ then we consider the two-sided ideal

$$T(E) = \{f \in F \mid f(e_1, \ldots, e_n) = 0, \text{ for all } e_1, \ldots, e_n \in E\}.$$

- The algebra $A = F/T(E)$ is called the *relatively free algebra in n variables which is defined by E*. The study of such structures is fundamental for the theory of *PI-algebras*.

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- Note that $T(E)$ is a *T-ideal*, that is, it is invariant under all endomorphisms of the algebra F.
- In particular, $T(E)$ is invariant under the action of GL_n on F by linear substitutions of variables. Then, A is a (polynomial) GL_n-module.
- In characteristic zero, we have that A is a direct sum of simple submodules

$$A = \bigoplus_{\lambda} m_\lambda W^\lambda$$

where the set $\{W^\lambda\}$ is parametrized by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $k \leq n$.
- The simple modules W^λ are finite-dimensional and multigraded with respect to $\deg(w) = (\alpha_1, \ldots, \alpha_n)$ where α_i is the number of times that the variable x_i occurs in the monomial (word) $w \in F$.

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- The multigraded Hilbert series of W^λ is called a *Schur function*. This is a *symmetric polynomial* that we denote by S_λ.
- Then, A is also multigraded and its multigraded Hilbert series $\mathrm{HS}(A)$ is a symmetric function admitting the Schur function decomposition
$$\mathrm{HS}(A) = \sum_\lambda m_\lambda S_\lambda.$$
- In other words, the GL_n-module structure (multiplicities m_λ) can be completely recovered by means of the (Schur function decomposition) of the multigraded Hilbert series of A.

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Example

- To compute such series, we start with a presentation of the algebra $A = F/T(E)$. The two-sided ideal $T(E) \subset F$ is generated by

$$[[x_i, x_j], x_k]\,,\ [x_i, x_j][x_k, x_l] + [x_i, x_k][x_j, x_l]$$

for all variables x_i, x_j, x_k, x_l.

- The leading monomial ideal I of $T(E)$ has been computed in

Drensky, Vesselin, —— ; Gröbner bases of ideals invariant under endomorphisms. J. Symbolic Comput., 41 (2006)

- For instance, for $n = 3$ the ideal I has the following *infinite* monomial basis

$$x^2y, x^2z, xy^2, xyz, xzy, xz^2, y^2z, yz^2, xyxy, xyxz,$$
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- We present now the method. The natural setting here are *monomial cyclic right modules*, that is, the free associative algebra modulo a monomial right ideal. In particular, one has the method for (monomial) algebras, (monomial) right modules, etc.
- Let $I \subset F$ be any monomial right ideal and put $C = F/I$ the corresponding monomial cyclic right module.
- Consider the multigraded right F-module homomorphism

$$\varphi : \bigoplus_{1 \leq i \leq n} F[-\bar{\theta}_i] \to C, \sum_i e_i f_i \mapsto \sum_i x_i f_i$$

where $\bar{\theta}_i = \deg(x_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$ in F and we define $\deg(1) = \bar{\theta}_i$ in $F[-\bar{\theta}_i]$.
- The image $\operatorname{Im} \varphi$ is the right submodule $B = \langle x_1, \ldots, x_n \rangle \subset C$. Then, the cokernel C/B is either 0 when $C = 0$ ($I = \langle 1 \rangle$) or it is isomorphic to the base field $\mathbb{K}$ otherwise.

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- What about the kernel $\mathrm{Ker}\,\varphi$?
- For any right ideal $I \subset F$ and each element $f \in F$ one defines

$$(I :_R f) = \{g \in F \mid fg \in I\}.$$

This is again a right ideal which is called the *colon right ideal* defined by I with respect to f.
- Clearly, we have $(I :_R f) = \langle 1 \rangle$ if and only if $f \in I$.
- If I is a monomial ideal and w is a monomial then $(I :_R w)$ is also a monomial ideal.
- Since we are in the monomial case, for the homomorphism φ one has that

$$\mathrm{Ker}\,\varphi = \bigoplus_{1 \leq i \leq n} (I :_R x_i)$$

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- By the exact sequence, one obtains the following key formula

$$\mathrm{HS}(C) = \sum_{1 \leq i \leq n} t_i \cdot \mathrm{HS}(C_{x_i}) + c(I)$$

- where, by definition, $c(I) = \dim_{\mathbb{K}}(C/B)$, that is

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- This formula suggests that one may compute a rational form for a noncommutative Hilbert series by successively applying the colon right ideal operation and solving over the rational function field $\mathbb{K}(t_1, \ldots, t_n)$ the corresponding linear equations.
- Similar ideas are used for computing commutative Hilbert series which are always rational functions.
- In the noncommutative case there are well-known counterexamples of irrational Hilbert series.
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Remark that it is essential to obtain a monomial basis of $I_{x_i} = (I :_R x_i)$ starting from the (possibly infinite) basis of I. In the papers, we provide general results for monomial right ideals. In particular, for two-sided ideals (algebras) we have the following

Proposition

Let $\{w_j\}$ be a basis of a monomial two-sided ideal I and consider a monomial w. Assume that $w \notin I$, that is, $(I :_R w) \neq \langle 1 \rangle$. For all j, we define the (finitely generated) monomial right ideal

$$R(w, w_j) = \langle v_{jk} \mid u_{jk} w_j = w v_{jk}, \deg(v_{jk}) < \deg(w_j) \rangle.$$

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Example

- Recall that for our example (relatively free algebra of the Grassman algebra in $n = 3$ variables) the leading monomial ideal I of $T(E)$ is infinitely generated by

$$x^2y, x^2z, xy^2, xyz, xzy, xz^2, y^2z, yz^2, xyxy, xyxz,$$
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- By denoting $C_x = F/I_x$ and $C_y = F/I_y$, we obtain a first linear equation over the field $\mathbb{Q}(t_1, t_2, t_3)$

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Example

- By continuing in this way, we obtain that the smallest stable set containing I (orbit) with respect to the colon right ideal operator, consists of 7 right ideals

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Example

- We have proved, in general, that such a linear system has a unique solution which provides the multigraded Hilbert series of all cyclic right modules which are defined by the right ideals in the orbit.
- In particular, for the algebra A of our example one obtains the symmetric rational function

$$\mathrm{HS}(A) = \frac{t_1 t_2 + t_1 t_3 + t_2 t_3 + 1}{(1 - t_1)(1 - t_2)(1 - t_3)}$$

- This formula specializes for $n = 3$ the general formula

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- Consider the two-sided ideal $B = \langle x_1, \ldots, x_n \rangle$ and its power B^{d+1} ($d \geq 0$) which is generated by all monomials of total degree $d + 1$. We define the *d-th truncation of A* as the finite-dimensional algebra $A^{(d)} = F/I^{(d)}$ where $I^{(d)} = I + B^{d+1}$.
- We can compute the multigraded Hilbert series (in fact polynomial) $\mathrm{HS}(A^{(d)})$ by our methods which are in this case (finite dimensional algebra) especially fast.
- In fact, we characterize when an algebra has *finite dimension* by the *nilpotency* of a matrix A_I which defines the linear system obtained by our method. Namely, for the graded (univariate) Hilbert series, this system corresponds to a matrix equation

$$(E - t \cdot A_I)\mathbf{H} = \mathbf{C}_I$$

where E is the identity matrix, $\mathbf{C}_I = (1, \ldots, 1)^t$ and $\mathbf{H}$ is the column vector of the unknown (non-zero) Hilbert series.

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- From the nilpotency of A_l it follows that this matrix can be reduced to a strictly upper triangular form which implies fast linear solving.
- We remark that truncation is also an essential tool when the Hilbert series of an algebra is *irrational*. In fact, it provides a polynomial (Taylor) approximation of this unknown function.
- Then, for the multigraded case we can combine our (fast) method for Hilbert series with (fast) polynomial Schur function decomposition.
- For the considered example, we compute the truncated Hilbert series up to degree $d = 10$ in *33 milliseconds* and its Schur function decomposition in *380 milliseconds*.

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Example

- The result of our computation for the relatively free algebra of the Grassmann algebra ($n = 3, d = 10$) is

$$\mathrm{HS}(A^{(10)}) = \sum_{0 \leq k \leq 10} \sum_{\substack{p+q=k,\\ 0 \leq q \leq 2}} S_{(p,1^q)}$$

- This computational result agrees with the general formula

$$\mathrm{HS}(A) = \sum_{k \geq 0} \sum_{\substack{p+q=k,\\ 0 \leq q \leq n-1}} S_{(p,1^q)}.$$

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- If $A = F/I$ is a monomial algebra (or cyclic right module), we want now to characterize when the ideal I admits a finite orbit $\mathcal{O}_I$ and hence A has a rational Hilbert series.
- For this task, we need the concept of *regular language*. It comes from formal language theory, a branch of theoretical computer science.
- If $X = \{x_1, \ldots, x_n\}$ is the set of variables of F, one denotes by X^* the set of monomials (words) of F. A *formal language* is by definition any subset $L \subset X^*$.
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- Given two languages L, L', one defines their set-theoretic *union* $L \cup L'$ and their *product* $L \cdot L' = \{ww' \mid w \in L, w' \in L'\}$.
- For any $d \geq 0$, we have also the power $L^d = \{w_1 \cdots w_d \mid w_i \in L\}$ the *star operation* $L^* = \bigcup_{d \geq 0} L^d$.
- The union, the product and the star operation are called the *rational operations* over the languages.

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Example

- Recall that the leading monomial ideal correspoding to the considered relatively free algebra $A = F/T(E)$ is the (infinitely generated) two-sided ideal

$$I = \langle x^2y, x^2z, xy^2, xyz, xzy, xz^2, y^2z, yz^2, xyxy, xyxz, xzxy, xzxz, yzyz\rangle + \langle yzy^dxy, yzy^dxz \mid d \geq 0\rangle.$$

- If $X = \{x, y, z\}$ then the language $L = I \cap X^*$ of the monomials of I is regular because it can be written as

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Theorem

The orbit $\mathcal{O}_I$ is finite if and only if the corresponding language $L = I \cap X^*$ is regular.

- Formal languages are important in theoretical computer science because they correspond to *models of computations*, like automata, grammars, Turing machines, etc.
- Regular languanges correspond to easiest models (Chomsky hierarchy) which are *finite automata*.
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Assume the orbit $\mathcal{O}_I$ is a finite set. The representation $w \mapsto T_w$ where $T_w(J) = (J :_R w)$ for any monomial right ideal $J \in \mathcal{O}_I$ defines a minimal automata corresponding to the regular language $L = I \cap X^*$.

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- To have an idea how efficient is this method, I provide here computing data for a non-trivial example.
- We consider the universal enveloping algebra of the free metabelian Lie algebra with $n = 6$ generators, that is, $A = F/I$ where I is the GL_6-invariant ideal of $F = \mathbb{K}\langle x_1, \ldots, x_6 \rangle$ generated by

$$[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}]] \quad (1 \leq i_1, i_2, i_3, i_4 \leq 6)$$

- We compute the truncation at degree $d = 10$ of both graded and multigraded Hilbert series in *13 seconds*.
- The number of right ideals in the corresponding orbit is 365. The number of monomials in the multigraded Hilbert series (in fact polynomial) is 8008.
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$$
\begin{gathered}
1+\ S[1]+\ S[1,1]+\ S[2]+\ S[1,1,1]+2\,S[2,1]+\ S[3]+\ S[1,1,1,1]+2\,S[2,1,1]+2\,S[2,2]\\
+3\,S[3,1]+\ S[4]+\ S[1,1,1,1,1]+2\,S[2,1,1,1]+3\,S[2,2,1]+4\,S[3,1,1]+5\,S[3,2]+4\,S[4,1]\\
+\ S[5]+\ S[1,1,1,1,1,1]+2\,S[2,1,1,1,1]+3\,S[2,2,1,1]+3\,S[2,2,2]+5\,S[3,1,1,1]\\
+10\,S[3,2,1]+5\,S[3,3]+7\,S[4,1,1]+9\,S[4,2]+5\,S[5,1]+\ S[6]+2\,S[2,1,1,1,1,1]\\
+3\,S[2,2,1,1,1]+4\,S[2,2,2,1]+5\,S[3,1,1,1,1]+13\,S[3,2,1,1]+11\,S[3,2,2]+13\,S[3,3,1]\\
+10\,S[4,1,1,1]+23\,S[4,2,1]+14\,S[4,3]+11\,S[5,1,1]+14\,S[5,2]+6\,S[6,1]+\ S[7]\\
+3\,S[2,2,1,1,1,1]+4\,S[2,2,2,1,1]+3\,S[2,2,2,2]+5\,S[3,1,1,1,1,1]+14\,S[3,2,1,1,1]\\
+20\,S[3,2,2,1]+20\,S[3,3,1,1]+21\,S[3,3,2]+11\,S[4,1,1,1,1]+37\,S[4,2,1,1]+30\,S[4,2,2]\\
+45\,S[4,3,1]+14\,S[4,4]+18\,S[5,1,1,1]+44\,S[5,2,1]+28\,S[5,3]+16\,S[6,1,1]+20\,S[6,2]\\
+7\,S[7,1]+\ S[8]+4\,S[2,2,2,1,1,1]+4\,S[2,2,2,2,1]+14\,S[3,2,1,1,1,1]+25\,S[3,2,2,1,1]\\
+17\,S[3,2,2,2]+24\,S[3,3,1,1,1]+48\,S[3,3,2,1]+19\,S[3,3,3]+12\,S[4,1,1,1,1,1]\\
+46\,S[4,2,1,1,1]+70\,S[4,2,2,1]+84\,S[4,3,1,1]+86\,S[4,3,2]+54\,S[4,4,1]+23\,S[5,1,1,1,1]\\
+84\,S[5,2,1,1]+67\,S[5,2,2]+108\,S[5,3,1]+42\,S[5,4]+30\,S[6,1,1,1]+75\,S[6,2,1]+48\,S[6,3]\\
+22\,S[7,1,1]+27\,S[7,2]+8\,S[8,1]+\ S[9]+4\,S[2,2,2,2,1,1]+3\,S[2,2,2,2,2]+26\,S[3,2,2,1,1,1]\\
+30\,S[3,2,2,2,1]+25\,S[3,3,1,1,1,1]+69\,S[3,3,2,1,1]+50\,S[3,3,2,2]+55\,S[3,3,3,1]\\
+50\,S[4,2,1,1,1,1]+103\,S[4,2,2,1,1]+69\,S[4,2,2,2]+117\,S[4,3,1,1,1]+238\,S[4,3,2,1]\\
+94\,S[4,3,3]+117\,S[4,4,1,1]+127\,S[4,4,2]+27\,S[5,1,1,1,1,1]+120\,S[5,2,1,1,1]\\
+186\,S[5,2,2,1]+237\,S[5,3,1,1]+238\,S[5,3,2]+190\,S[5,4,1]+42\,S[5,5]+44\,S[6,1,1,1,1]\\
+166\,S[6,2,1,1]+131\,S[6,2,2]+217\,S[6,3,1]+90\,S[6,4]+47\,S[7,1,1,1]+118\,S[7,2,1]\\
+75\,S[7,3]+29\,S[8,1,1]+35\,S[8,2]+9\,S[9,1]+\ S[10]
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- We have implemented all these methods in the kernel of the computer algebra system SINGULAR

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- We remark finally that SINGULAR is a *commutative algebra system* which means that our implementation relies on the notion of *letterplace correspondence* which is able to translate noncommutative structures into commutative ones.
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