Systems of Equations in Nilpotent Groups

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Initial question

Is $\mathscr{D}(G)$ decidable, for G nilpotent?

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- $[x, y] = x^{-1}y^{-1}xy$.
- $G_1(G) = G,\ G_2 = [G, G], \ldots,\ G_k = [G_{k-1}, G]$.
- G is *nilpotent* if $G_k = 1$ for some k.

Equations in nilpotent groups - known results

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Single equations:

- Single equations are decidable in some 2-step nilpotent groups. (Duchin, Liang, Shapiro, 2014).
- Hence single equations and systems of equations are 'essentially different' in nilpotent groups. This is not the case for $(\mathbb{Z}, +, \cdot)$, for example.

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Definition

- $R \subseteq G$ (or $\subseteq G^n$) is *e-definable in* G if "$g \in R$" can be expressed as a system over G:

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- An operation $\odot$ in R is *e-definable* in G if "$\vec{z} = \vec{x} \odot \vec{y}$," can be expressed as a system over G.

Definable and interpretable structures

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In both cases:

- $\mathscr{D}(R, +, \cdot)$ is reducible to $\mathscr{D}(G)$.
- If $R = \mathbb{Z}$, then $\mathscr{D}(G)$ is undecidable, by the negative answer to Hilbert's 10th problem.

Rings of algebraic integers

Theorem (G., Miasnikov, Ovchinnikov)

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Conjecture

$\mathscr{D}(G)$ is undecidable for any f.g. non-virtually abelian nilpotent group G.

Where does the ring $\mathcal{O}$ come from?

- The following is a bilinear map between abelian groups:

$$\begin{aligned}[\cdot,\cdot] : G/G' \times G/G' &\to G'/G_3 \\ [gG', hG'] &\mapsto [g,h]G_3\end{aligned}$$

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- The set of endomorphisms $End(A)$ of an abelian group A, with the operations $+$ and $\circ$, forms a non-commutative ring.
- $\mathcal{O}$ is constructed using subrings of $End(G/G')$ and $End(G'/G_3)$ (the maximal ring of scalars of $[\cdot,\cdot]$).

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- There are no known examples of (non-virtually abelian) metabelian groups with decidable DP...
- ... but they all have undecidable first-order theory (Noskov 1983).

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  - G contains a f.g. non-virtually abelian nilpotent group.
- Examples: Subgroups of $GL(n, K)$ for K a ring of algebraic integers. Discrete solvable subgroups of $GL(n, \mathbb{R})$.

Grazie per l'attenzione!

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