On coverings of word values in profinite groups

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If $\{G_i\}_{i\in I}$ is a covering of X and all the G_i's have some property, what can we infer on the structure of $\langle X\rangle$?

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If the family is finite and $X = G$, it turns out that a lot can be said.

Baer

A group can be covered by finitely many abelian subgroups if and only if it is central-by-finite (the centre has finite index).

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Baire Category Theorem

If a locally compact Hausdorff space is a union of countably many closed subsets, then at least one of them has non-empty interior.

It follows that

if a profinite group (or a closed subset of it) is covered by countably many subgroups, then at least one of the covering subgroups is open.

BAER

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COVERINGS OF G

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ACCIARRI, SHUMYATSKY 2014

A profinite group can be covered by countably many procyclic subgroups if and only if it is finite-by-procyclic.

Shumyatsky 2016

For a profinite group G the following conditions are equivalent:

1. G is covered by countably many abelian subgroups;
2. G is central-by-finite;
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For a profinite group G the following conditions are equivalent:

1. G is covered by countably many nilpotent subgroups;
2. G is finite-by-nilpotent;
3. There exists a positive integer m such that $Z_m(G)$ is open.

Here $Z_m(G)$ denotes the m-th term of the upper central series of G.

Let $\mathcal{C}$ be one of the following classes of groups.

- The class of pronilpotent groups;
- The class of locally nilpotent groups;
- The class of strongly locally nilpotent groups.

Recall that a group G is locally nilpotent if all finitely generated subgroups of G are nilpotent.
Following Shalev, we say that a group G is strongly locally nilpotent if it belongs to a locally nilpotent variety of groups. This means that, for some function f and for all positive integers d, every d-generated subgroup of G is nilpotent of class at most $f(d)$.

D, Morigi, Shumyatsky 2017

For a profinite group G the following conditions are equivalent:

1. G is covered by countably many $\mathcal{C}$-subgroups;
2. G is covered by finitely many $\mathcal{C}$-subgroups;
3. G is finite-by-$\mathcal{C}$;

Only if

A word w on n variables is an element of the free group F with free generators $x_1, \ldots, x_n$ and we think of w as a function $w : G^n \mapsto G$.
If the set of w-values in a group G can be covered by some subgroups, one could hope to get some information on the structure of the verbal subgroup $w(G)$ generated by all w-values.

COVERINGS OF w-VALUES

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If the set of w-values in a group G can be covered by some subgroups, one could hope to get some information on the structure of the verbal subgroup $w(G)$ generated by all w-values.

- If $w = x$, then $w(G) = G$ and we have all the results on coverings of G.
- If $w = [x, y]$, then $w(G) = G'$ and we have several results on coverings of commutators.

COVERINGS OF COMMUTATORS

ACCIARRI-SHUMYATSKY 2014, SHUMYATSKY 2015, D-MORIGI-SHUMYATSKY 2017

Let G be a profinite group.

1. The set of commutators is covered by countably many procyclic subgroups if and only if G' is finite-by-procyclic.
2. The set of commutators is covered by countably many nilpotent subgroups if and only if G' is finite-by-nilpotent.
3. The set of commutators is covered by countably many $\mathcal{C}$-subgroups if and only if G' is finite-by-$\mathcal{C}$.

Recall that $\mathcal{C}$ is the class of pronilpotent groups, or the class of locally nilpotent groups, or the class of strongly locally nilpotent groups.

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Moreover, even the case of abstract group and finite coverings is open.

Question

Let G be an abstract group in which commutators are covered by finitely many abelian (nilpotent) subgroups. Is G' necessarily finite-by-abelian (finite-by-nilpotent, resp.)?

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For example, the word

$$[[x_1, x_2], [x_3, x_4, x_5], x_6]$$

is a multilinear commutator while the Engel word $[x, y, y, y]$ is not. An important family of multilinear commutator words is formed by the derived words δ_k, on 2^k variables, defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the the k-th term of the derived series of G.

A group is called locally finite if each of its finitely generated subgroups is finite. A group is said to be of finite rank r if each subgroup of G can be generated by at most r elements.

D, Morigi, Shumyatsky 2015

Let w be a multilinear commutator word and G a profinite group. The set of w-values in G is covered by countably many locally finite (finite rank) subgroups if and only if $w(G)$ is locally finite (finite rank, resp.).

The case of finite coverings was solved by Acciarri and Shumyatsky (2011).

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The case of finite coverings was solved by Acciarri and Shumyatsky (2011).
Recall that multilinear commutator words are concise in the class of all groups (Wilson, 1974) that is $w(G)$ is finite if and only if the set of w-values in G is finite.

D, Morigi, Shumyatsky 2016

Let w be a multilinear commutator word and G a profinite group. Then $w(G)$ is finite if and only if the set of w-values in G is countable.

We recently generalised the results on coverings of G (or commutators) by $\mathcal{C}$-subgroups, to coverings of the w-values where w is an arbitrary multilinear commutator word.
Let now $\mathcal{C}$ be one of the following classes of groups.

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Main Theorem (D, Morigi, Shumyatsky)

Let w be a multilinear commutator word and G a profinite group. The set of w-values in G is covered by countably many finite-by-$\mathcal{C}$ subgroups if and only if $w(G)$ is finite-by-$\mathcal{C}$.

Since a profinite group is finite-by-$\mathcal{C}$ if and only if it is covered by finitely many $\mathcal{C}$-subgroups, we have the following corollary of the main theorem:

Corollary (D, Morigi, Shumyatsky 2017)

Let w be a multilinear commutator word and G a profinite group. The following statements are equivalent.

1. The verbal subgroup $w(G)$ is finite-by-$\mathcal{C}$;
2. The set of w-values in G is covered by countably many $\mathcal{C}$-subgroups;
3. The set of w-values in G is covered by finitely many $\mathcal{C}$-subgroups.

Unsurprisingly, the proof of the main theorem is much more complicated than the proofs in the case where $w = x$ or $w = [x, y]$; it relies on a development of some combinatorial techniques for handling multilinear commutator words which were introduced by Fernàndez-Alcober and Morigi and used in some of our previous works.

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The easiest result to be shown, which might be of independent interest, is the following:

Let G be a group and let w be a multilinear commutator of weight n. Assume that H is a normal subgroup of G such that for some elements $a_1, \dots, a_n \in G$ the identity $w(a_1H, \dots, a_nH) = 1$ holds. Then $w(H) = 1$.

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Moreover:

If G is soluble, then $w(G)$ has a finite abelian series of normal subgroups, each of which is generated by w-values.

We use the following result as a tool to overcome the fact that the class $\mathcal{C}$ is not extension closed:

Let w be a word and let G be a profinite group in which the set of w-values is covered by countably many $\mathcal{C}$-subgroups. Suppose that N is a normal open $\mathcal{C}$-subgroup of G. If x is a w-value, then the subgroup $\langle N, x \rangle$ is in $\mathcal{C}$.

We remark that the classes $\mathcal{C}$ are closed under the product of two normal subgroups.

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We remark that the classes $\mathcal{C}$ are closed under the product of two normal subgroups.
Moreover,

G has a normal open subgroup H such that $w(H)$ is virtually-$\mathcal{C}$.

So, at least in the case where G is soluble, the proof follows by the fact that $w(G)$ has a finite abelian series of normal subgroups, each of which is generated by w-values: we "enlarge" $w(H)$, adding appropriate w-values, until we reach $w(G)$.

To "reduce" to the soluble case, we use this well-known result:

If w is a multilinear commutator word on n variables and G a group, then each δ_n-value in G is a w-value.

Hence the set of δ_n-values of G is covered by countably many $\mathcal{C}$-subgroups, and we prove that $G^{(2n)}$ is virtually-$\mathcal{C}$.
Then we prove that in this situation, $G^{(2n)}$ is actually finite-by-$\mathcal{C}$, and we start to "enlarge" this subgroup, until we reach $w(G)$.

A finite-by-$\mathcal{C}$ profinite group G is covered by finitely many $\mathcal{C}$-subgroups.

Indeed, assume that G is a finite-by-$\mathcal{C}$ profinite group, let K be a normal finite subgroup such that G/K is in $\mathcal{C}$ and let N be an open normal subgroup of G such that $N \cap K = 1$. As N has finite index in G, there are only finitely many subgroups of the form $\langle a, N\rangle$ and they cover G. Therefore it is sufficient to prove that for every $a \in G$ the subgroup $\langle a, N\rangle$ is in $\mathcal{C}$. This is clear when $\mathcal{C}$ is the class of pronilpotent or locally nilpotent groups.
So now we will assume that G/K is n-Engel and we want to prove that for every $a \in G$ the subgroup $\langle a, N\rangle$ is n-Engel. If $x, y \in \langle a, N\rangle$, we see that $[x,_n y] \in K$ and $[x,_n y] \in \langle a, N\rangle' \leq N$. Hence $[x,_n y] = 1$.

Return