ON COVERINGS OF WORD VALUES
IN PROFINITE GROUPS

Eloisa Detomi

Università di Padova, Italy

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A covering of a subset $X$ of a group $G$ is a family $\{G_i\}_{i \in I}$ of subgroups of $G$ such that $X \subseteq \bigcup_{i \in I} G_i$. 

If the family is finite and $X = G$, it turns out that a lot can be said. A group can be covered by finitely many abelian subgroups if and only if it is central-by-finite (the centre has finite index).
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**Question**

If $\{G_i\}_{i \in I}$ is a covering of $X$ and all the $G_i$’s have some property, what can we infer on the structure of $\langle X \rangle$?
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**Baer**

A group can be covered by **finitely** many **abelian** subgroups if and only if it is central-by-finite (the centre has finite index).
An important tool for dealing with problems of this kind is B. H. Neumann’s Lemma:

If \( \{G_i\}_{i \in I} \) is a finite covering of \( G \), then \( G \) is actually covered by the subgroups \( G_i \) of finite index.

If \( G \) is a profinite group, we can deal with coverings containing countably many subgroups. Recall that a profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.

THEOREM
If a locally compact Hausdorff space is a union of countably many closed subsets, then at least one of them has non-empty interior. It follows that if a profinite group (or a closed subset of it) is covered by countably many subgroups, then at least one of the covering subgroups is open.
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**Baire Category Theorem**

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A group covered by finitely many cyclic subgroups is either cyclic or finite.
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A profinite group can be covered by countably many procyclic subgroups if and only if it is finite-by-procyclic.
For a profinite group $G$ the following conditions are equivalent:

1. $G$ is covered by countably many abelian subgroups;
2. $G$ is central-by-finite;
3. $G$ is finite-by-abelian.

Here $Z_m(G)$ denotes the $m$-th term of the upper central series of $G$. 
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For a profinite group $G$ the following conditions are equivalent:

1. $G$ is covered by countably many nilpotent subgroups;
2. $G$ is finite-by-nilpotent;
3. There exists a positive integer $m$ such that $Z_m(G)$ is open.

Here $Z_m(G)$ denotes the $m$-th term of the upper central series of $G$. 
Let $C$ be one of the following classes of groups.
- The class of pronilpotent groups;
- The class of locally nilpotent groups;
- The class of strongly locally nilpotent groups.

Recall that a group $G$ is locally nilpotent if all finitely generated subgroups of $G$ are nilpotent.

Following Shalev, we say that a group $G$ is strongly locally nilpotent if it belongs to a locally nilpotent variety of groups. This means that, for some function $f$ and for all positive integers $d$, every $d$-generated subgroup of $G$ is nilpotent of class at most $f(d)$.

**D, Morigi, Shumyatsky 2017**

For a profinite group $G$ the following conditions are equivalent:
1. $G$ is covered by *countably* many $C$-subgroups;
2. $G$ is covered by *finitely* many $C$-subgroups;
3. $G$ is finite-by-$C$;
A word $w$ on $n$ variables is an element of the free group $F$ with free generators $x_1, \ldots, x_n$ and we think of $w$ as a function $w : G^n \to G$. If the set of $w$-values in a group $G$ can be covered by some subgroups, one could hope to get some information on the structure of the verbal subgroup $w(G)$ generated by all $w$-values.
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- If $w = x$, then $w(G) = G$ and we have all the results on coverings of $G$.
- If $w = [x, y]$, then $w(G) = G'$ and we have several results on coverings of commutators.
Let $G$ be a profinite group.

1. The set of commutators is covered by countably many procyclic subgroups if and only if $G'$ is finite-by-procyclic.

2. The set of commutators is covered by countably many nilpotent subgroups if and only if $G'$ is finite-by-nilpotent.

3. The set of commutators is covered by countably many $C$-subgroups if and only if $G'$ is finite-by-$C$.

Recall that $C$ is the class of pronilpotent groups, or the class of locally nilpotent groups, or the class of strongly locally nilpotent groups.
The case where commutators are covered by countably many abelian subgroups is still open.

**Question**

Let $G$ be a profinite group in which commutators are covered by countably many abelian subgroups. Is $G'$ necessarily finite-by-abelian?

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**Question**

Let $G$ be an abstract group in which commutators are covered by finitely many abelian (nilpotent) subgroups. Is $G'$ necessarily finite-by-abelian (finite-by-nilpotent, resp.)?
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For example, the word

$$[[x_1, x_2], [x_3, x_4, x_5], x_6]$$

is a multilinear commutator while the Engel word $[x, y, y, y]$ is not. An important family of multilinear commutator words is formed by the derived words $\delta_k$, on $2^k$ variables, defined recursively by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \ldots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \ldots, x_{2^k})].$$

Of course $\delta_k(G) = G^{(k)}$ is the the $k$-th term of the derived series of $G$. 
A group is called **locally finite** if each of its finitely generated subgroups is finite. A group is said to be of **finite rank** $r$ if each subgroup of $G$ can be generated by at most $r$ elements.

**D, Morigi, Shumyatsky 2015**

Let $w$ be a multilinear commutator word and $G$ a profinite group. The set of $w$-values in $G$ is covered by **countably many** **locally finite** (finite rank) subgroups if and only if $w(G)$ is locally finite (finite rank, resp.).

The case of finite coverings was solved by Acciarri and Shumyatsky (2011).
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Recall that multilinear commutator words are concise in the class of all groups (Wilson, 1974) that is $w(G)$ is finite if and only if the set of $w$-values in $G$ is finite.

### D, Morigi, Shumyatsky 2016

Let $w$ be a multilinear commutator word and $G$ a profinite group. Then $w(G)$ is **finite** if and only if the set of $w$-values in $G$ is countable.
We recently generalised the results on coverings of $G$ (or commutators) by $C$-subgroups, to coverings of the $w$-values where $w$ is an arbitrary multilinear commutator word.

Let now $C$ be one of the following classes of groups.

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**Main Theorem (D, Morigi, Shumyatsky)**

Let $w$ be a multilinear commutator word and $G$ a profinite group. The set of $w$-values in $G$ is covered by countably many finite-by-$C$ subgroups if and only if $w(G)$ is finite-by-$C$. 
Since a profinite group is finite-by-$C$ if and only if it is covered by finitely many $C$-subgroups, we have the following corollary of the main theorem:

**Corollary (D, Morigi, Shumyatsky 2017)**

Let $w$ be a multilinear commutator word and $G$ a profinite group. The following statements are equivalent.

1. The verbal subgroup $w(G)$ is finite-by-$C$;
2. The set of $w$-values in $G$ is covered by countably many $C$-subgroups;
3. The set of $w$-values in $G$ is covered by finitely many $C$-subgroups.
Unsurprisingly, the proof of the main theorem is much more complicated than the proofs in the case where \( w = x \) or \( w = [x, y] \); it relies on a development of some combinatorial techniques for handling multilinear commutator words which were introduced by Fernàndez-Alcober and Morigi and used in some of our previous works.

The easiest result to be shown, which might be of independent interest, is the following:

Let \( G \) be a group and let \( w \) be a multilinear commutator of weight \( n \). Assume that \( H \) is a normal subgroup of \( G \) such that for some elements \( a_1, \ldots, a_n \in G \) the identity \( w(a_1H, \ldots, a_nH) = 1 \) holds. Then \( w(H) = 1 \).

Moreover:

If \( G \) is soluble, then \( w(G) \) has a finite abelian series of normal subgroups, each of which is generated by \( w \)-values.
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Moreover:

If \( G \) is soluble, then \( w(G) \) has a finite abelian series of normal subgroups, each of which is generated by \( w \)-values.
We use the following result as a tool to overcome the fact that the class $C$ is not extension closed:

Let $w$ be a word and let $G$ be a profinite group in which the set of $w$-values is covered by countably many $C$-subgroups. Suppose that $N$ is a normal open $C$-subgroup of $G$. If $x$ is a $w$-value, then the subgroup $\langle N, x \rangle$ is in $C$.

We remark that the classes $C$ are closed under the product of two normal subgroups.
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We remark that the classes $C$ are closed under the product of two normal subgroups.
Moreover,

$G$ has a normal open subgroup $H$ such that $w(H)$ is virtually-$C$.

So, at least in the case where $G$ is soluble, the proof follows by the fact that $w(G)$ has a finite abelian series of normal subgroups, each of which is generated by $w$-values: we "enlarge" $w(H)$, adding appropriate $w$-values, until we reach $w(G)$. 
To "reduce" to the soluble case, we use this well-known result:

If $w$ is a multilinear commutator word on $n$ variables and $G$ a group, then each $\delta_n$-value in $G$ is a $w$-value.

Hence the set of $\delta_n$-values of $G$ is covered by countably many $C$-subgroups, and we prove that $G^{(2n)}$ is virtually-$C$. Then we prove that in this situation, $G^{(2n)}$ is actually finite-by-$C$, and we start to "enlarge" this subgroup, until we reach $w(G)$. 


A finite-by-$\mathcal{C}$ profinite group $G$ is covered by finitely many $\mathcal{C}$-subgroups.

Indeed, assume that $G$ is a finite-by-$\mathcal{C}$ profinite group, let $K$ be a normal finite subgroup such that $G/K$ is in $\mathcal{C}$ and let $N$ be an open normal subgroup of $G$ such that $N \cap K = 1$. As $N$ has finite index in $G$, there are only finitely many subgroups of the form $\langle a, N \rangle$ and they cover $G$. Therefore it is sufficient to prove that for every $a \in G$ the subgroup $\langle a, N \rangle$ is in $\mathcal{C}$. This is clear when $\mathcal{C}$ is the class of pronilpotent or locally nilpotent groups.

So now we will assume that $G/K$ is $n$-Engel and we want to prove that for every $a \in G$ the subgroup $\langle a, N \rangle$ is $n$-Engel. If $x, y \in \langle a, N \rangle$, we see that $[x, _n y] \in K$ and $[x, _n y] \in \langle a, N \rangle' \leq N$. Hence $[x, _n y] = 1$. 

Return