CHARACTERIZING THE GROUP OF COLEMAN AUTOMORPHISMS

Arne Van Antwerpen

September 4, 2017

VRIJE UNIVERSITEIT BRUSSEL

DEFINITIONS

Definition

Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. If for any prime p dividing the order of G and any Sylow p-subgroup P of G, there exists a $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$, then σ is said to be a Coleman automorphism.

DEFINITIONS

Definition

Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. If for any prime p dividing the order of G and any Sylow p-subgroup P of G, there exists a $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$, then σ is said to be a Coleman automorphism.

Denote $\mathrm{Aut}_{col}(G)$ for the set of Coleman automorphisms, $\mathrm{Inn}(G)$ for the set of inner automorphisms and set

$$\mathrm{Out}_{col}(G) = \mathrm{Aut}_{col}(G)/\mathrm{Inn}(G).$$

ELEMENTARY RESULTS

Theorem (Hertweck and Kimmerle)

Let G be a finite group. The prime divisors of $|\mathrm{Aut}_{col}(G)|$ are also prime divisors of $|G|$.

ELEMENTARY RESULTS

Theorem (Hertweck and Kimmerle)

Let G be a finite group. The prime divisors of $|\mathrm{Aut}_{col}(G)|$ are also prime divisors of $|G|$.

Lemma

Let N be a normal subgroup of a finite group G. If $\sigma \in \mathrm{Aut}_{col}(G)$, then $\sigma|_N \in \mathrm{Aut}(N)$.

NORMALIZER PROBLEM

Let G be a group and R a ring (denote $U(RG)$ for the units of RG), then we clearly have that

$$GC_{U(RG)}(G) \subseteq N_{U(RG)}(G)$$

NORMALIZER PROBLEM

Let G be a group and R a ring (denote $U(RG)$ for the units of RG), then we clearly have that

$$GC_{U(RG)}(G) \subseteq N_{U(RG)}(G)$$

Is this an equality?

$$N_{U(RG)}(G) = GC_{U(RG)}(G)$$

Of special interest: $R = \mathbb{Z}$

SETUP FOR REFORMULATION

If $u \in N_{U(RG)}(G)$, then u induces an automorphism of G:

$$\begin{aligned} \varphi_u : G &\to G \\ g &\mapsto u^{-1}gu \end{aligned}$$

SETUP FOR REFORMULATION

If $u \in N_{U(RG)}(G)$, then u induces an automorphism of G:

$$\begin{aligned} \varphi_u : G &\to G \\ g &\mapsto u^{-1} g u \end{aligned}$$

Denote $\mathrm{Aut}_U(G; R)$ for the group of these automorphisms ($\mathrm{Aut}_U(G)$ if $R = \mathbb{Z}$) and $\mathrm{Out}_U(G; R) = \mathrm{Aut}_U(G; R)/\mathrm{Inn}(G)$

REFORMULATION

Theorem (Jackowski and Marciniak)

G a finite group, R a commutative ring. TFAE

1. $N_{U(RG)}(G) = GC_{U(RG)}(G)$
2. $\mathrm{Aut}_U(G;R) = \mathrm{Inn}(G)$

REFORMULATION

Theorem (Jackowski and Marciniak)

G a finite group, R a commutative ring. TFAE

1. $N_{U(RG)}(G) = GC_{U(RG)}(G)$
2. $\mathrm{Aut}_U(G;R) = \mathrm{Inn}(G)$

Theorem

Let G be a finite group. Then,

$$\mathrm{Aut}_U(G) \subseteq \mathrm{Aut}_{col}(G)$$

SADLY, A COUNTEREXAMPLE

Theorem (Hertweck)

There exists a finite metabelian group G of order $2^{25}97^2$ with

$$\mathrm{Aut}_{U(RG)}(G) \neq \mathrm{Inn}(G).$$

SOME MOTIVATION

Theorem (Dade)

Let G be a finite group. Then, $\mathrm{Out}_{col}(G)$ *is a nilpotent group.*

SOME MOTIVATION

Theorem (Dade)

Let G be a finite group. Then, $\mathrm{Out}_{col}(G)$ *is a nilpotent group.*

Dade conjectured that $\mathrm{Out}_{col}(G)$ is even an abelian group, due to the following theorem.

SOME MOTIVATION

Theorem (Dade)

Let G be a finite group. Then, $\mathrm{Out}_{col}(G)$ *is a nilpotent group.*

Dade conjectured that $\mathrm{Out}_{col}(G)$ is even an abelian group, due to the following theorem.

Theorem (Dade)

Let A be a finite abelian group. Then there exists a finite metabelian group G such that $\mathrm{Out}_{col}(G) \cong A$.

DADE´S CONJECTURE REVISITED

Using the structure of the groups, Hertweck and Kimmerle showed that $\mathrm{Out}_{col}(G) = 1$ for several classes of finite groups.

DADE´S CONJECTURE REVISITED

Using the structure of the groups, Hertweck and Kimmerle showed that $\mathrm{Out}_{col}(G) = 1$ for several classes of finite groups.

Theorem (Hertweck, Kimmerle)

Let G be a finite group. Then $\mathrm{Out}_{col}(G)$ is a finite abelian group.

INTEREST IN CHARACTERIZING AUT$_{COL}(G)$

1. As an interesting subclass of Aut(G)

INTEREST IN CHARACTERIZING $\mathrm{AUT}_{COL}(G)$

1. As an interesting subclass of $\mathrm{Aut}(G)$
2. To study $\mathrm{Out}_{col}(G)$

INTEREST IN CHARACTERIZING AUT$_{COL}(G)$

1. As an interesting subclass of $\mathrm{Aut}(G)$
2. To study $\mathrm{Out}_{col}(G)$
3. To find classes of finite groups where another counterexample to the Normalizer problem can be found

GENERALIZED DIHEDRAL GROUPS

Theorem

Let A be an abelian group and $C_2 = \langle \eta \rangle$. Denote $D = A \rtimes \eta$, where η acts by inversion. Then,

$$\mathrm{Aut}_{col}(D) = \mathrm{Inn}(D) \rtimes K,$$

where K is an elementary abelian 2-group.

DEFINING THE OUTER AUTOMORPHISMS

Definition

Let A be a finite abelian group and $C_q = \langle x \rangle$ a cyclic group of order $q = p^r$, with p a prime number. Write $A = A_{p_1} \times \ldots \times A_{p_n}$, its Sylow decomposition. Then, for any $1 \leq i \leq n$ and non-negative integers k_i, define the group homomorphisms

$$\phi_{k_1 \ldots k_n} : A * C_q \to A * C_q$$

where, for $g_i \in A_{p_i}$,

$$g_i \mapsto x^{-k_i} g_i x^{k_i} \text{ and } x \mapsto x$$

ABELIAN-BY-CYCLIC GROUPS

Theorem

Let $G = A \rtimes C_q$, with notation as before. For any $1 \leq i \leq n$, let r_i be the smallest positive integer such that $\mathrm{conj}(x^{r_i})|_{A_{p_i}} = \mathrm{id}|_{A_{p_i}}$. Assume that the Sylow subgroups are ordered such that $r_1 \leq ... \leq r_n$. Then,

$$\mathrm{Aut}_{col}(G) = \mathrm{Inn}(G) \rtimes K,$$

with

$$K = \left\{ \Phi_{k_1 \dots k_{n-1} 0} \mid k_i \in \mathbb{Z}_{r_i} \right\}.$$

Thus,

$$K \cong C_{r_1} \times ... \times C_{r_{n-1}}.$$

NILPOTENT-BY-CYCLIC GROUPS

Theorem

Let G be a finite nilpotent-by-(p-power-cyclic) group. Then, under very technical specifications

$$\mathrm{Aut}_{col}(G) \cong \mathrm{Inn}(G) \rtimes K.$$

QUESTIONS OF HERTWECK AND KIMMERLE

Questions (Hertweck and Kimmerle)

1. *Is* $\mathrm{Out}_{col}(G)$ *a* p'*-group if* G *does not have* C_p *as a chief factor?*

QUESTIONS OF HERTWECK AND KIMMERLE

Questions (Hertweck and Kimmerle)

1. *Is* $\mathrm{Out}_{col}(G)$ *a* p'*-group if* G *does not have* C_p *as a chief factor?*
2. *Is* $\mathrm{Out}_{col}(G)$ *trivial if* $O_{p'}(G)$ *is trivial?*

QUESTIONS OF HERTWECK AND KIMMERLE

Questions (Hertweck and Kimmerle)

1. *Is* $\mathrm{Out}_{col}(G)$ *a* p'*-group if* G *does not have* C_p *as a chief factor?*
2. *Is* $\mathrm{Out}_{col}(G)$ *trivial if* $O_{p'}(G)$ *is trivial?*
3. *Is* $\mathrm{Out}_{col}(G)$ *trivial if* G *has a unique minimal non-trivial normal subgroup?*

QUESTIONS OF HERTWECK AND KIMMERLE

Questions (Hertweck and Kimmerle)

1. *Is* $\mathrm{Out}_{col}(G)$ *a* p'*-group if* G *does not have* C_p *as a chief factor?*
2. *Is* $\mathrm{Out}_{col}(G)$ *trivial if* $O_{p'}(G)$ *is trivial?*
3. *Is* $\mathrm{Out}_{col}(G)$ *trivial if* G *has a unique minimal non-trivial normal subgroup?*

Partial Answer (Hertweck, Kimmerle)

Besides giving several conditions, all three statements hold if G *is assumed to be a* p*-constrained group.*

NEW ANSWERS

Partial Answers (Van Antwerpen)

1. *No new result.*

NEW ANSWERS

Partial Answers (Van Antwerpen)

1. *No new result.*
2. *True, if $O_p(G) = O_{p'}(G) = 1$, where p is an odd prime and the order of every direct component of $E(G)$ is divisible by p.*

NEW ANSWERS

Partial Answers (Van Antwerpen)

1. *No new result.*
2. *True, if $O_p(G) = O_{p'}(G) = 1$, where p is an odd prime and the order of every direct component of $E(G)$ is divisible by p.*
3. *True, if the unique minimal non-trivial normal subgroup is non-abelian. True, if question 2 has a positive answer.*

FURTHER RESEARCH

Inkling of Idea

In case G has a unique minimal non-trivial normal subgroup N, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.

FURTHER RESEARCH

Inkling of Idea

In case G has a unique minimal non-trivial normal subgroup N, which we may assume to be abelian, we may be able to use the classification of the simple groups and the list of all Schur multipliers to give a technical proof.

Related Question

Gross conjectured that for a finite group G with $O_{p'}(G) = 1$ for some odd prime p, p-central automorphisms of p-power order are inner. Hertweck and Kimmerle believed this is possible using the classification of Schur multipliers.

REFERENCES

1. M. Hertweck and W. Kimmerle. Coleman automorphisms of finite groups.
2. S.Jackowski and Z.Marciniak. Group automorphisms inducing the identity map on cohomology.
3. F. Gross. Automorphisms which centralize a Sylow *p*-subgroup.
4. E.C. Dade. Locally trivial outer automorphisms of finite groups.
5. A. Van Antwerpen. Coleman automorphisms of finite groups and their minimal normal subgroups. Arxiv preprint.