On large orbits of actions of finite groups. Applications

Dedicated to the memory of
Professor James Clark Beidleman

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Introduction

All sets, groups, modules and fields are finite.

Introduction

- The solution of the $k(GV)$-problem ($k(GV) \leq |V|$) depends on the existence of regular orbits.

P. Schmid.
The solution of the $k(GV)$-problem, volume 4 of *ICP Advanced Texts in Mathematics*.
Imperial College Press, London, 2007.

Introduction

Definition

If G acts on $\Omega \neq \emptyset$, $w \in \Omega$ is in a regular orbit if $\mathrm{C}_G(w) = \{g \in G : wg = w\} = 1$, that is, the orbit of w is as large as possible and has size $|G|$.

Introduction

Natural question: existence of regular orbits.
Interesting case: $\Omega = V$ a G-module.

Introduction

Our motivation: open questions about intersections of prefrattini subgroups and system normalisers of soluble groups raised by Kamornikov, Shemetkov and Vasil′ev in Kourovka Notebook.

Introduction

Definition

Let k be a positive integer. A 3-tuple (G, X, Y) is said to be a k-conjugate system if G is a group, X, Y are subgroups of G with $Y = \mathrm{Core}_G(X)$, and there exist k elements $g_1, \dots, g_k$ such that $Y = \bigcap_{i=1}^{k} X^{g_i}$.

Introduction

Known results about conjugate systems

Theorem (Dolfi)

If π is a set of primes and G is a π-soluble group, then $(G, H, O_\pi(G))$ is a 3-conjugate system, where H is a Hall π-subgroup of G.

S. Dolfi.
Large orbits in coprime actions of solvable groups.
Trans. Amer. Math. Soc., 360(1):135–152, 2008.

Introduction
Known results about conjugate systems

Theorem (Dolfi)

If π is a set of primes and G is a π-soluble group, then $(G, H, O_\pi(G))$ is a 3-conjugate system, where H is a Hall π-subgroup of G.

Particular cases:

- Passman ($|\pi| = 1$)

D. S. Passman.
Groups with normal, solvable Hall p'-subgroups.
Trans. Amer. Math. Soc., 123(1):99–111, 1966.

Introduction

Known results about conjugate systems

Theorem (Dolfi)

If π is a set of primes and G is a π-soluble group, then $(G, H, O_{\pi}(G))$ is a 3-conjugate system, where H is a Hall π-subgroup of G.

Particular cases:

- Zenkov (H nilpotent)

V. I. Zenkov.
The structure of intersections of nilpotent π-subgroups in π-solvable finite groups.
Siberian Math. J., 34(4): 683–687 ,1993.

Introduction
Known results about conjugate systems

Mann pointed out that the results of Passman imply that $(G, I, \mathrm{F}(G))$ is a 3-conjugate system, where $\mathrm{F}(G)$ is the Fitting subgroup of a soluble group G and I is a nilpotent injector of G.

A. Mann.
The intersection of Sylow subgroups.
Proc. Amer. Math. Soc., 53(2):262–264, 1975.
Addendum: ibidem Vol. 62, No. 1, p. 188 (1977).

Introduction

Two questions about conjugate systems

Problem (Kamornikov, Problem 17.55)

Does there exist an absolute constant k such that $(G, H, \Phi(G))$ is a k-conjugate system for any soluble group G and any prefrattini subgroup H of G?

V. D. Mazurov and E. I. Khukhro, editors.
Unsolved problems in Group Theory: The Kourovka Notebook.
Russian Academy of Sciences, Siberian Branch, Institute of Mathematics, Novosibirsk, Russia, 17 edition, 2010.

Introduction

Two questions about conjugate systems

Problem (Shemetkov and Vasil′ev, Problem 17.39)

Is there a positive integer k such that $(G, D, Z_\infty(G))$ is a k-conjugate system for any soluble group G and any system normaliser D of G? What is the least number with this property?

V. D. Mazurov and E. I. Khukhro, editors.
Unsolved problems in Group Theory: The Kourovka Notebook.
Russian Academy of Sciences, Siberian Branch, Institute of Mathematics, Novosibirsk, Russia, 17 edition, 2010.

Introduction

Two questions about conjugate systems

These kind of questions can be reduced to a problem about regular orbits in faithful actions of groups.

Assume we want to prove that (G, X, Y) is a k-conjugate system by induction on the order of the soluble group G. Then we may assume that $Y = 1$, $k > 1$ and, in many cases, that $G = NX$, where N is self-centralising minimal normal subgroup of G which is complemented in G by the core-free maximal subgroup X. Note that $X \cap X^n = \mathrm{C}_X(n)$. Therefore, if the natural action of X on $N \oplus \cdots^{(k-1)} \cdots \oplus N$ has a regular orbit, then there exist $n_1, \ldots n_{k-1} \in N$ such that $\mathrm{C}_X(n_1) \cap \ldots \mathrm{C}_X(n_{k-1}) = 1$ and so $X \cap X^{n_1} \ldots X^{n_{k-1}} = 1$.

Introduction
Gluck's conjecture

- $\mathrm{Irr}(G)$: set of all irreducible complex characters of G.
- $b(G) = \max\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$: largest irreducible (complex) character degree of G.

Gluck showed that if G is soluble, then $|G : \mathrm{F}(G)| \leq b(G)^{13/2}$ and conjectures:

Conjecture (Gluck, 1985)

$|G : \mathrm{F}(G)| \leq b(G)^2$.

D. Gluck.
The largest irreducible character degree of a finite group.
Canad. J. Math., 37(3):442–415, 1985.

Introduction

Gluck's conjecture

Gluck's conjecture is still open and has been studied extensively.

Gluck's strategy: consider the action of $G/\operatorname{F}(G)$ on the faithful and completely reducible $G/\operatorname{F}(G)$-module V of all linear characters of the section $\operatorname{F}(G)/\Phi(G)$. We have that large orbits of $G/\operatorname{F}(G)$ on V give large character degrees.

To prove Gluck's conjecture in this way, it is enough to prove that if V is a faithful completely reducible G-module, then there exists an orbit in V of length at least $\sqrt{|G|}$. We could get such an orbit by means of a regular orbit of G on $V \oplus V$.

Introduction
Known results about regular orbits

- Espuelas proved that if G is a group of odd order and V is a faithful and completely reducible G-module of odd characteristic, then G has a regular orbit on $V \oplus V$.

A. Espuelas.
Large character degrees of groups of odd order.
Illinois J. Math., 35(3):499–505, 1991.

Introduction
Known results about regular orbits

- Dolfi and Jabara extended Espuelas’ result to the case where the Sylow 2-subgroups of the semidirect product $[V]G$ of V and the soluble group G are abelian.
- Yang proved that the same is true if 3 does not divide the order of the soluble group G.

S. Dolfi and E. Jabara.
Large character degrees of solvable groups with abelian Sylow 2-subgroups.
J. Algebra, 313(2):687–694, 2007.

Y. Yang.
Orbits of the actions of finite solvable groups.
J. Algebra, 321:2012–2021, 2009.

Introduction

Known results about regular orbits

Dolfi, reproving a result of Seress, proved that any soluble group G has a regular orbit on $V \oplus V \oplus V$ and if either $(|V|, |G|) = 1$ or G is of odd order, then G has also a regular orbit on $V \oplus V$.

S. Dolfi.
Large orbits in coprime actions of solvable groups.
Trans. Amer. Math. Soc., 360(1):135–152, 2008.

Á. Seress.
The minimal base size of primitive solvable permutation groups.
J. London Math. Soc., 53(2):243–255, 2006.

Introduction

Known results about regular orbits

- A result of Wolf shows that a similar result holds if G is supersoluble (see also Moretó and Wolf for an improved result when G is nilpotent).

T .Wolf.
Large orbits of supersolvable linear groups.
J. Algebra, 215:235–247, 1999.

A. Moretó and T. R. Wolf.
Orbit sizes, character degrees and Sylow subgroups.
Adv. Math., 184(1):18–36, 2004.
Erratum: ibid., no. 2, page 409.

Introduction

Known results about regular orbits

- More recently, Yang (2014) extends some of these results to the case when H is a subgroup of the soluble group G by proving that if V is a faithful completely reducible G-module (possibly of mixed characteristic) and if either H is nilpotent or 3 does not divide the order of H, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $[V]H$ are abelian, then H has at least two regular orbits on $V \oplus V$.

Y. Yang.
Large orbits of subgroups of solvable linear groups.
Israel J. Math., 199(1):345–362, 2014.

Our main theorems

Theorem (with Meng and Esteban-Romero)

Let G be a soluble group and let V be a faithful completely reducible G-module (possibly of mixed characteristic). Suppose that H is a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H has at least three regular orbits on $V \oplus V$.

- Recall that if G and X are groups, then G is said to be X-free if X cannot be obtained as a quotient of a subgroup of G; $\Gamma(2^3)$ denotes the semilinear group of the Galois field of 2^3 elements.
- The S_4-free hypothesis in the above theorem is not superfluous (Dolfi and Jabara, 2007).

Our main theorems

Corollary (Yang)

Let G be a soluble group acting completely reducibly and faithfully on a module V. Suppose that H is a subgroup of G. If H is nilpotent or $3 \nmid |H|$, then H has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product VH are abelian, then H has at least two regular orbits on $V \oplus V$.

Y. Yang.
Large orbits of subgroups of solvable linear groups.
Israel J. Math., 199(1):345–362, 2014.

Our main theorems

Corollary (Dolfi)

Let G be a soluble group and V be a faithful completely reducible G-module. Suppose that $(|G|, |V|) = 1$. Then G has at least two regular orbits on $V \oplus V$.

S. Dolfi.
Large orbits in coprime actions of solvable groups.
Trans. Amer. Math. Soc., 360(1):135–152, 2008.

Our main theorems

Theorem (with Meng)

Let G be a soluble group acting completely reducibly and faithfully on a module V. If H is a supersoluble subgroup of G, then H has at a regular orbit on $V \oplus V$.

Our main theorems

Theorem (with Meng and Esteban-Romero)

Let G be a soluble group satisfying one of the following conditions:

1. *G is S_4-free;*
2. *$G/\operatorname{F}(G)$ is S_4-free and $\operatorname{F}(G)$ is of odd order;*
3. *$G/\operatorname{F}(G)$ is S_3-free.*

Then Gluck's conjecture is true for G.

Our main theorems

Corollary (Dolfi and Jabara; Cossey, Halasi, Maróti, Nguyen)

Let G be a soluble group. If either the Sylow 2-subgroups of G are abelian or $|G/\operatorname{F}(G)|$ is not divisible by 6, then Gluck's conjecture is true for G.

S. Dolfi and E. Jabara.
Large character degrees of solvable groups with abelian Sylow 2-subgroups.
J. Algebra, 313(2):687–694, 2007.

J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen.
On a conjecture of Gluck.
Math. Z., 279:1067–1080, 2015.

System normalisers and prefrattini subgroups

Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and such that every group G has an smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of G and denoted by $G^{\mathfrak{F}}$. A maximal subgroup M of a group G containing $G^{\mathfrak{F}}$ is called $\mathfrak{F}$-normal in G; otherwise, M is said to be $\mathfrak{F}$-abnormal.

System normalisers and prefrattini subgroups

We say that $\mathfrak{F}$ is saturated if it is closed under Frattini extensions. In such case, by a well-known theorem of Gaschütz-Lubeseder-Schmid , there exists a collection of formations $\mathrm{F}(p) \subseteq \mathfrak{F}$, one for each prime p, such that $\mathfrak{F}$ coincides with the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in \mathrm{F}(p)$ for all primes p dividing $|H/K|$. In this case, we say that H/K is $\mathfrak{F}$-central in G and $\mathfrak{F}$ is locally defined by the $\mathrm{F}(p)$. H/K is called $\mathfrak{F}$-eccentric if it is not $\mathfrak{F}$-central.

System normalisers and prefrattini subgroups

Note that a chief factor H/K supplemented by a maximal subgroup M is $\mathfrak{F}$-central in G if and only if M is $\mathfrak{F}$-normal in G. Every group G has a largest normal subgroup such that every chief factor of G below it is $\mathfrak{F}$-central in G. This subgroup is called the $\mathfrak{F}$-hypercentre of G and it is denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$

System normalisers and prefrattini subgroups

Let Σ be a Hall system of the soluble group G.
Let S^p be the p-complement of G contained in Σ, and denote by $W^p(G)$ the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p ($W^p(G) = G$ if the set of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p is empty).

System normalisers and prefrattini subgroups

Then $W(G, \Sigma, \mathfrak{F}) = \bigcap_{p \in \pi(G)} W^p(G)$ is called the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ.

The prefrattini subgroups of G form a characteristic class of G-conjugate subgroups and they were introduced by Gaschütz and Hawkes.

A. Ballester-Bolinches and L. M. Ezquerro.
Classes of Finite Groups, volume 584 of *Mathematics and Its Applications*.
Springer, Dordrecht, 2006.

K. Doerk and T. Hawkes.
Finite Soluble Groups, volume 4 of *De Gruyter Expositions in Mathematics*.
Walter de Gruyter, Berlin, New York, 1992.

System normalisers and prefrattini subgroups

The intersection $\mathrm{L}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-abnormal maximal subgroups of a soluble group G is the core of every $\mathfrak{F}$-prefrattini subgroup of G and $\mathrm{L}_{\mathfrak{F}}(G)/\Phi(G) = \mathrm{Z}_{\mathfrak{F}}(G/\Phi(G))$ for every group G.

System normalisers and prefrattini subgroups

Theorem (with Cossey, Kamornikov and Meng)

Let $\mathfrak{F}$ be a saturated formation and let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system.

System normalisers and prefrattini subgroups

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G.
It is a characteristic nilpotent subgroup of G that was introduced by Gaschütz (1953).
If $\mathfrak{F}$ is the trivial formation, then $\mathrm{L}_{\mathfrak{F}}(G) = \Phi(G)$, the Frattini subgroup of G.

W. Gaschütz.
Über die Φ-Untergruppe endlicher Gruppen.
Math. Z., 58:160–170, 1953.

System normalisers and prefrattini subgroups

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G.

Corollary (Kamornikov)

If G is soluble and H is an $\mathfrak{N}$-prefrattini subgroup of G, then $(G, H, \mathrm{L}(G))$ is a 3-conjugate system.

S. F. Kamornikov.
One characterization of the Gaschütz subgroup of a finite soluble group.
Fundamentalnaya i Prikladnaya Mathematica, 20:65–75, 2015.
Russian.

System normalisers and prefrattini subgroups

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G.

Corollary (Kamornikov)

If G is soluble and H is a prefrattini subgroup of G, then $(G, H, \Phi(G))$ is a 3-conjugate system.

S.F. Kamornikov.
Intersections of prefrattini subgroups in finite soluble groups.
Int. J. Group Theory, 6(2): 1–5, 2017.

System normalisers and prefrattini subgroups

Let $F(p)$ be a particular family of formations locally defining $\mathfrak{F}$ and such that $F(p) \subseteq \mathfrak{F}$ for all primes p.
Let $\pi = \{p : F(p) \text{ non-empty}\}$. For an arbitrary soluble group G and a Hall system Σ of G, choose for any prime p, the p-complement $K^p = S^p \cap G^{F(p)}$ of the $F(p)$-residual $G^{F(p)}$ of G, where S^p is the p-complement of G in Σ. Then $D_{\mathfrak{F}}(\Sigma) = G_\pi \cap (\bigcap_{p\in\pi} N_G(K^p))$, where G_π is the Hall π-subgroup of G in Σ, is the $\mathfrak{F}$-normaliser of G associated to Σ.
The $\mathfrak{F}$-normalisers of G are a characteristic class of G-conjugate subgroups. There were introduced by Carter and Hawkes and coincide with the classical system normalisers of Hall when $\mathfrak{F}$ is the formation of all nilpotent groups.
If D is an $\mathfrak{F}$-normaliser of G, then $\mathrm{Core}_G(D) = Z_{\mathfrak{F}}(G)$.

System normalisers and prefrattini subgroups

Theorem (with Cossey, Kamornikov and Meng)

Let $\mathfrak{F}$ be a saturated formation and let D be an $\mathfrak{F}$-normaliser of a soluble group G such that $\Phi(G) = 1$. Then $(G, D, Z_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, D, Z_{\mathfrak{F}}(G))$ is a 3-conjugate system.

System normalisers and prefrattini subgroups

Corollary (with Cossey, Kamornikov and Meng)

Let G be a soluble group with $\Phi(G) = 1$. If D is a system normaliser of G, then $(G, D, Z_\infty(G))$ is a 3-conjugate system.

System normalisers and prefrattini subgroups

Example

Let D be the dihedral group of order 8. Then D has an irreducible and faithful module V of dimension 2 over the field of 3-elements such that $C_D(v) \neq 1$ for all $v \in V$. Let $G = V \rtimes D$ be the corresponding semidirect product. Then D is a system normaliser of G and $Z_\infty(G) = 1$. $D \cap D^v = C_D(v) \neq 1$ for all $v \in V$. Hence $(G, D, Z_\infty(G))$ is not a 2-conjugate system.

System normalisers and prefrattini subgroups

A class of groups $\mathfrak{F}$ is said to be a Fitting class if $\mathfrak{F}$ is a class under taking subnormal subgroups and such that every group G has a largest normal $\mathfrak{F}$-subgroup called $\mathfrak{F}$-radical and denoted by $G_{\mathfrak{F}}$. Every soluble group G has a conjugacy class of subgroups, called $\mathfrak{F}$-injectors, which are defined to be those subgroups I of G such that if S is a subnormal subgroup of G, then $I \cap S$ is $\mathfrak{F}$-maximal subgroup of S. Note that, in this case, $\mathrm{Core}_G(I) = G_{\mathfrak{F}}$.

System normalisers and prefrattini subgroups

Theorem (with Cossey, Kamornikov and Meng)

Let $\mathfrak{F}$ be a Fitting class and let I be an $\mathfrak{F}$-injector of a soluble group G. Then $(G, I, G_{\mathfrak{F}})$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, I, G_{\mathfrak{F}})$ is a 3-conjugate system.

System normalisers and prefrattini subgroups

Corollary (Passman-Mann)

If G is soluble and I is a nilpotent injector of G, then $(G, I, \mathrm{F}(G))$ is a 3-conjugate system.