On large orbits of actions of finite groups.
Applications

Dedicated to the memory of Professor James Clark Beidleman

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All sets, groups, modules and fields are finite.
The solution of the $k(GV)$-problem ($k(GV) \leq |V|$) depends on the existence of regular orbits.

P. Schmid. 

*The solution of the $k(GV)$-problem*, volume 4 of *ICP Advanced Texts in Mathematics*. 
Definition

If $G$ acts on $\Omega \neq \emptyset$, $w \in \Omega$ is in a regular orbit if $C_G(w) = \{g \in G : wg = w\} = 1$, that is, the orbit of $w$ is as large as possible and has size $|G|$. 
Natural question: existence of regular orbits.
Interesting case: $\Omega = V$ a $G$-module.
Our motivation: open questions about intersections of prefrattini subgroups and system normalisers of soluble groups raised by Kamornikov, Shemetkov and Vasil’ev in Kourovka Notebook.
Definition

Let $k$ be a positive integer. A 3-tuple $(G, X, Y)$ is said to be a $k$-conjugate system if $G$ is a group, $X, Y$ are subgroups of $G$ with $Y = \text{Core}_G(X)$, and there exist $k$ elements $g_1, \ldots, g_k$ such that $Y = \bigcap_{i=1}^k X^{g_i}$. 
Introduction

Known results about conjugate systems

Theorem (Dolfi)

If $\pi$ is a set of primes and $G$ is a $\pi$-soluble group, then $(G, H, O_{\pi}(G))$ is a 3-conjugate system, where $H$ is a Hall $\pi$-subgroup of $G$.

S. Dolfi.

Large orbits in coprime actions of solvable groups.

Theorem (Dolfi)

If $\pi$ is a set of primes and $G$ is a $\pi$-soluble group, then $(G, H, O_\pi(G))$ is a 3-conjugate system, where $H$ is a Hall $\pi$-subgroup of $G$.

Particular cases:

- Passman ($|\pi| = 1$)

D. S. Passman.
Groups with normal, solvable Hall $p'$-subgroups. 
Theorem (Dolfi)

If $\pi$ is a set of primes and $G$ is a $\pi$-soluble group, then $(G, H, O_{\pi}(G))$ is a 3-conjugate system, where $H$ is a Hall $\pi$-subgroup of $G$.

Particular cases:

- Zenkov ($H$ nilpotent)

Mann pointed out that the results of Passman imply that $(G, I, F(G))$ is a 3-conjugate system, where $F(G)$ is the Fitting subgroup of a soluble group $G$ and $I$ is a nilpotent injector of $G$.

A. Mann.

The intersection of Sylow subgroups.


Problem (Kamornikov, Problem 17.55)

Does there exist an absolute constant $k$ such that $(G, H, \Phi(G))$ is a $k$-conjugate system for any soluble group $G$ and any prefrattini subgroup $H$ of $G$?

V. D. Mazurov and E. I. Khukhro, editors.

*Unsolved problems in Group Theory: The Kourovka Notebook.*

Problem (Shemetkov and Vasil’ev, Problem 17.39)

Is there a positive integer $k$ such that $(G, D, Z_\infty(G))$ is a $k$-conjugate system for any soluble group $G$ and any system normaliser $D$ of $G$? What is the least number with this property?

V. D. Mazurov and E. I. Khukhro, editors.  
Unsolved problems in Group Theory: The Kourovka Notebook.  
These kind of questions can be reduced to a problem about regular orbits in faithful actions of groups. Assume we want to prove that \((G, X, Y)\) is a \(k\)-conjugate system by induction on the order of the soluble group \(G\). Then we may assume that \(Y = 1\), \(k > 1\) and, in many cases, that \(G = NX\), where \(N\) is self-centralising minimal normal subgroup of \(G\) which is complemented in \(G\) by the core-free maximal subgroup \(X\). Note that \(X \cap X^n = C_X(n)\). Therefore, if the natural action of \(X\) on \(N \oplus \cdots \oplus N\) has a regular orbit, then there exist \(n_1, \ldots, n_{k-1} \in N\) such that \(C_X(n_1) \cap \cdots \cap C_X(n_{k-1}) = 1\) and so \(X \cap X^{n_1} \cdots X^{n_{k-1}} = 1\).
Introduction

Gluck’s conjecture

- \( \text{Irr}(G) \): set of all irreducible complex characters of \( G \).
- \( b(G) = \max\{\chi(1) \mid \chi \in \text{Irr}(G)\} \): largest irreducible (complex) character degree of \( G \).

Gluck showed that if \( G \) is soluble, then \( |G : F(G)| \leq b(G)^{13/2} \) and conjectures:

**Conjecture (Gluck, 1985)**

\[ |G : F(G)| \leq b(G)^2. \]

D. Gluck.
Gluck’s conjecture is still open and has been studied extensively. **Gluck’s strategy**: consider the action of $G/ F(G)$ on the faithful and completely reducible $G/ F(G)$-module $V$ of all linear characters of the section $F(G)/\Phi(G)$. We have that large orbits of $G/ F(G)$ on $V$ give large character degrees. To prove Gluck’s conjecture in this way, it is enough to prove that if $V$ is a faithful completely reducible $G$-module, then there exists an orbit in $V$ of length at least $\sqrt{|G|}$. We could get such an orbit by means of a regular orbit of $G$ on $V \oplus V$. 
Espuelas proved that if $G$ is a group of odd order and $V$ is a faithful and completely reducible $G$-module of odd characteristic, then $G$ has a regular orbit on $V \oplus V$.

A. Espuelas.
Large character degrees of groups of odd order.

Dolfi and Jabara extended Espuelas’ result to the case where the Sylow 2-subgroups of the semidirect product $[V]G$ of $V$ and the soluble group $G$ are abelian.

Yang proved that the same is true if 3 does not divide the order of the soluble group $G$.


Dolfi, reproving a result of Seress, proved that any soluble group $G$ has a regular orbit on $V \oplus V \oplus V$ and if either $(|V|, |G|) = 1$ or $G$ is of odd order, then $G$ has also a regular orbit on $V \oplus V$.

S. Dolfi.
Large orbits in coprime actions of solvable groups.

Á. Seress.
The minimal base size of primitive solvable permutation groups.
A result of Wolf shows that a similar result holds if $G$ is supersoluble (see also Moretó and Wolf for an improved result when $G$ is nilpotent).

- **T. Wolf.**
  Large orbits of supersolvable linear groups.  

- **A. Moretó and T. R. Wolf.**
  Orbit sizes, character degrees and Sylow subgroups.  
  Erratum: ibid., no. 2, page 409.
More recently, Yang (2014) extends some of these results to the case when $H$ is a subgroup of the soluble group $G$ by proving that if $V$ is a faithful completely reducible $G$-module (possibly of mixed characteristic) and if either $H$ is nilpotent or 3 does not divide the order of $H$, then $H$ has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $[V]H$ are abelian, then $H$ has at least two regular orbits on $V \oplus V$.

Y. Yang.

Our main theorems

Theorem (with Meng and Esteban-Romero)

Let $G$ be a soluble group and let $V$ be a faithful completely reducible $G$-module (possibly of mixed characteristic). Suppose that $H$ is a subgroup of $G$ such that the semidirect product $VH$ is $S_4$-free. Then $H$ has at least two regular orbits on $V \oplus V$. Furthermore, if $H$ is $\Gamma(2^3)$-free and $\text{SL}(2,3)$-free, then $H$ has at least three regular orbits on $V \oplus V$.

- Recall that if $G$ and $X$ are groups, then $G$ is said to be $X$-free if $X$ cannot be obtained as a quotient of a subgroup of $G$; $\Gamma(2^3)$ denotes the semilinear group of the Galois field of $2^3$ elements.
- The $S_4$-free hypothesis in the above theorem is not superfluous (Dolfi and Jabara, 2007).
Corollary (Yang)

Let $G$ be a soluble group acting completely reducibly and faithfully on a module $V$. Suppose that $H$ is a subgroup of $G$. If $H$ is nilpotent or $3 
mid |H|$, then $H$ has at least three regular orbits on $V \oplus V$. If the Sylow 2-subgroups of the semidirect product $VH$ are abelian, then $H$ has at least two regular orbits on $V \oplus V$.

Y. Yang.
Large orbits of subgroups of solvable linear groups.
Corollary (Dolfi)

Let $G$ be a soluble group and $V$ be a faithful completely reducible $G$-module. Suppose that $(|G|, |V|) = 1$. Then $G$ has at least two regular orbits on $V \oplus V$.

S. Dolfi.
Large orbits in coprime actions of solvable groups. 
Our main theorems

Theorem (with Meng)

Let $G$ be a soluble group acting completely reducibly and faithfully on a module $V$. If $H$ is a supersoluble subgroup of $G$, then $H$ has at a regular orbit on $V \oplus V$. 

Adolfo Ballester-Bolinches

Regular orbits of finite groups
Our main theorems

Theorem (with Meng and Esteban-Romero)

Let $G$ be a soluble group satisfying one of the following conditions:

1. $G$ is $S_4$-free;
2. $G/F(G)$ is $S_4$-free and $F(G)$ is of odd order;
3. $G/F(G)$ is $S_3$-free.

Then Gluck's conjecture is true for $G$. 
Corollary (Dolfi and Jabara; Cossey, Halasi, Maróti, Nguyen)

Let $G$ be a soluble group. If either the Sylow 2-subgroups of $G$ are abelian or $|G/F(G)|$ is not divisible by 6, then Gluck’s conjecture is true for $G$.


Recall that a formation is a class of groups $\mathcal{F}$ which is closed under taking epimorphic images and such that every group $G$ has an smallest normal subgroup with quotient in $\mathcal{F}$. This subgroup is called the $\mathcal{F}$-residual of $G$ and denoted by $G^{\mathcal{F}}$. A maximal subgroup $M$ of a group $G$ containing $G^{\mathcal{F}}$ is called $\mathcal{F}$-normal in $G$; otherwise, $M$ is said to be $\mathcal{F}$-abnormal.
We say that $\mathcal{F}$ is saturated if it is closed under Frattini extensions. In such case, by a well-known theorem of Gaschütz-Lubeseder-Schmid, there exists a collection of formations $F(p) \subseteq \mathcal{F}$, one for each prime $p$, such that $\mathcal{F}$ coincides with the class of all groups $G$ such that if $H/K$ is a chief factor of $G$, then $G/C_G(H/K) \in F(p)$ for all primes $p$ dividing $|H/K|$. In this case, we say that $H/K$ is $\mathcal{F}$-central in $G$ and $\mathcal{F}$ is locally defined by the $F(p)$. $H/K$ is called $\mathcal{F}$-eccentric if it is not $\mathcal{F}$-central.
Note that a chief factor $H/K$ supplemented by a maximal subgroup $M$ is $\mathcal{F}$-central in $G$ if and only if $M$ is $\mathcal{F}$-normal in $G$. Every group $G$ has a largest normal subgroup such that every chief factor of $G$ below it is $\mathcal{F}$-central in $G$. This subgroup is called the $\mathcal{F}$-hypercentre of $G$ and it is denoted by $Z_\mathcal{F}(G)$.
Let $\Sigma$ be a Hall system of the soluble group $G$. Let $S^p$ be the $p$-complement of $G$ contained in $\Sigma$, and denote by $W^p(G)$ the intersection of all $\mathcal{F}$-abnormal maximal subgroups of $G$ containing $S^p$ ($W^p(G) = G$ if the set of all $\mathcal{F}$-abnormal maximal subgroups of $G$ containing $S^p$ is empty).
System normalisers and prefrattini subgroups

Then $W(G, \Sigma, \mathcal{F}) = \bigcap_{p \in \pi(G)} W^p(G)$ is called the $\mathcal{F}$-prefrattini subgroup of $G$ associated to $\Sigma$. The prefrattini subgroups of $G$ form a characteristic class of $G$-conjugate subgroups and they were introduced by Gaschütz and Hawkes.


The intersection $L_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-abnormal maximal subgroups of a soluble group $G$ is the core of every $\mathfrak{F}$-prefrattini subgroup of $G$ and $L_{\mathfrak{F}}(G)/\Phi(G) = Z_{\mathfrak{F}}(G/\Phi(G))$ for every group $G$. 
Theorem (with Cossey, Kamornikov and Meng)

Let \( \mathcal{F} \) be a saturated formation and let \( H \) be an \( \mathcal{F} \)-prefrattini subgroup of a soluble group \( G \). Then \( (G, H, L_{\mathcal{F}}(G)) \) is a 4-conjugate system. Furthermore, if either \( G \) is \( S_4 \)-free or \( \mathcal{F} \) is composed of \( S_3 \)-free groups, then \( (G, H, L_{\mathcal{F}}(G)) \) is a 3-conjugate system.
If $\mathcal{F} = \mathcal{N}$, the formation of all nilpotent groups, then $L_{\mathcal{F}}(G) = L(G)$ is the intersection of all self-normalising maximal subgroups of $G$.
It is a characteristic nilpotent subgroup of $G$ that was introduced by Gaschütz (1953).
If $\mathcal{F}$ is the trivial formation, then $L_{\mathcal{F}}(G) = \Phi(G)$, the Frattini subgroup of $G$.

W. Gaschütz.
Über die $\Phi$-Untergruppe endlicher Gruppen.
If $\mathcal{F} = \mathcal{N}$, the formation of all nilpotent groups, then $L_\mathcal{F}(G) = L(G)$ is the intersection of all self-normalising maximal subgroups of $G$.

**Corollary (Kamornikov)**

*If $G$ is soluble and $H$ is an $\mathcal{N}$-prefrattini subgroup of $G$, then $(G, H, L(G))$ is a 3-conjugate system.*

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*S. F. Kamornikov.*

One characterization of the Gaschütz subgroup of a finite soluble group.


Russian.
If $\mathcal{F} = \mathcal{N}$, the formation of all nilpotent groups, then $L_\mathcal{F}(G) = L(G)$ is the intersection of all self-normalising maximal subgroups of $G$.

**Corollary (Kamornikov)**

*If $G$ is soluble and $H$ is a prefrattini subgroup of $G$, then $(G, H, \Phi(G))$ is a 3-conjugate system.*

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S.F. Kamornikov.

Intersections of prefrattini subgroups in finite soluble groups.

Let $F(p)$ be a particular family of formations locally defining $\mathcal{F}$ and such that $F(p) \subseteq \mathcal{F}$ for all primes $p$.
Let $\pi = \{p : F(p) \text{ non-empty}\}$. For an arbitrary soluble group $G$ and a Hall system $\Sigma$ of $G$, choose for any prime $p$, the $p$-complement $K^p = S^p \cap G^{F(p)}$ of the $F(p)$-residual $G^{F(p)}$ of $G$, where $S^p$ is the $p$-complement of $G$ in $\Sigma$. Then
\[ D_{\mathcal{F}}(\Sigma) = G_\pi \cap \left( \bigcap_{p \in \pi} N_G(K^p) \right), \]
where $G_\pi$ is the Hall $\pi$-subgroup of $G$ in $\Sigma$, is the $\mathcal{F}$-normaliser of $G$ associated to $\Sigma$.
The $\mathcal{F}$-normalisers of $G$ are a characteristic class of $G$-conjugate subgroups. There were introduced by Carter and Hawkes and coincide with the classical system normalisers of Hall when $\mathcal{F}$ is the formation of all nilpotent groups.
If $D$ is an $\mathcal{F}$-normaliser of $G$, then $\text{Core}_G(D) = Z_{\mathcal{F}}(G)$. 
Theorem (with Cossey, Kamornikov and Meng)

Let $\mathfrak{F}$ be a saturated formation and let $D$ be an $\mathfrak{F}$-normaliser of a soluble group $G$ such that $\Phi(G) = 1$. Then $(G, D, Z_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either $G$ is $S_4$-free or $\mathfrak{F}$ is composed of $S_3$-free groups, then $(G, D, Z_{\mathfrak{F}}(G))$ is a 3-conjugate system.
Corollary (with Cossey, Kamornikov and Meng)

Let $G$ be a soluble group with $\Phi(G) = 1$. If $D$ is a system normaliser of $G$, then $(G, D, Z_\infty(G))$ is a 3-conjugate system.
Example

Let $D$ be the dihedral group of order 8. Then $D$ has an irreducible and faithful module $V$ of dimension 2 over the field of 3-elements such that $C_D(v) \neq 1$ for all $v \in V$. Let $G = V \rtimes D$ be the corresponding semidirect product. Then $D$ is a system normaliser of $G$ and $Z_\infty(G) = 1$. $D \cap D^v = C_D(v) \neq 1$ for all $v \in V$. Hence $(G, D, Z_\infty(G))$ is not a 2-conjugate system.
A class of groups \( \mathcal{F} \) is said to be a **Fitting class** if \( \mathcal{F} \) is a class under taking subnormal subgroups and such that every group \( G \) has a largest normal \( \mathcal{F} \)-subgroup called \( \mathcal{F} \)-**radical** and denoted by \( G_{\mathcal{F}} \). Every soluble group \( G \) has a conjugacy class of subgroups, called \( \mathcal{F} \)-**injectors**, which are defined to be those subgroups \( I \) of \( G \) such that if \( S \) is a subnormal subgroup of \( G \), then \( I \cap S \) is \( \mathcal{F} \)-maximal subgroup of \( S \). Note that, in this case, \( \text{Core}_G(I) = G_{\mathcal{F}} \).
Theorem (with Cossey, Kamornikov and Meng)

Let $\mathcal{F}$ be a Fitting class and let $I$ be an $\mathcal{F}$-injector of a soluble group $G$. Then $(G, I, G_{\mathcal{F}})$ is a 4-conjugate system. Furthermore, if either $G$ is $S_4$-free or $\mathcal{F}$ is composed of $S_3$-free groups, then $(G, I, G_{\mathcal{F}})$ is a 3-conjugate system.
Corollary (Passman-Mann)

If $G$ is soluble and $I$ is a nilpotent injector of $G$, then $(G, I, F(G))$ is a 3-conjugate system.