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The Survey on Infinite Groups: a Guide to Some Classical Areas

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The book is a kind of guide that makes it possible to obtain information about basis concepts and results of some important parts of infinite group theory.

The book can be useful for algebraists intending to take a closer look at the theory of infinite groups in order to use obtained results there. It will also be useful for students who aim to become acquainted with the subject.

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Книга є своєрідним довідником, який дає можливість отримати інформацію про базові поняття та результати деяких важливих розділів теорії нескінченних груп.

Книга може бути корисною для алгебраїстів, що мають намір ближче познайомитись з теорією нескінченних груп та використовувати отримані в ній результати. Вона буде також корисною для аспірантів та студентів, які ставлять за мету познайомитись з цією тематикою.

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Infinite groups is my favorite subject

The Survey On Infinite Groups: a Guide to Some Classical Areas

L.A. Kurdachenko, A.A. Pypka, N.N. Semko, I.Ya. Subbotin

Contents

Introduction

The main goal of this book is to collect and demonstrate in a logical way some key concepts and present – day results from some classical areas of infinite group theory. While our main targeted audience is graduate students and beginner researchers in the area, we think that this book will be someway as beneficial for the experts as a reference book as well. The quite long and completed list of journal articles and books, and the direct references on the proper sources from this list, which every single mentioned result is supplied, justify this statement. We would like to add that this book would be also useful for those mathematicians who do not know or just forgot old proven results. It could prevent some repeated researches in the area. For example, we can mention some recent reinventions of groups with one or two classes of conjugate non – normal subgroups, despite the fact that those classes were completely described by O.Yu. Schmidt about eight decades ago.

We were thinking about the possible structure of this book for a long time. We did not want to write a textbook on group theory. A textbook requires presentation of a set of core results, foundations of the theory. Besides these core foundations, the choice of the selected material is usually defined and formed by the authors' interests and tests. Moreover, the textbook does not involve deep penetration into the material. This deep penetration is achieved in the monographs. However, it is limited breadth essentially. In group theory, we can observe a wide variety of methods and approaches, which can be considered as common. Perhaps therein lies its charm. Therefore, a presentation of methodology, techniques, and constructions occupies a large portion of each monograph on groups. It cools the enthusiasm of many people who are not experts in the area. Therefore, they cannot reach useful and potentially helpful to their research results. Therefore, we have chosen the survey form. Each of us has repeatedly had the opportunity to evaluate the benefits of survey articles, and it persuaded us to the choice of such form of the book. The book consists only of definitions and theorems. We just made direct references on the articles, surveys and books in where one can easily find a proof of the particular result. This book could be considered as a kind of guidebook on some central provinces of Infinite Group Theory Country. We chose those provinces based on our test and experience. Quite significant summative authors experience partially justifies the choice of contents.

The book consists of five chapters.

There are numbers of books, in which the properties of various classes of generalized soluble groups received complete and detailed consideration. In Chapter 1 «*On some classes of generalized soluble groups*», we focus only on those classes of generalized soluble groups that are proven to be the most popular and useful.

In Chapter 2 «*On some finiteness conditions in infinite groups*», we consider different classical finiteness conditions and related key concepts.

The presence of certain numerical invariants associated with an algebraic structure often opens the doors to deep and detailed study of this structure. A striking example of it is vector spaces. They possess an essential numerical characteristic, which is called dimension. Groups are also associated with numerical characteristics, some of which in one way or another are analogs of the concept of dimension of a vector space. These numerical invariants are not universal, and since they are introduced and employed in specific classes of groups, their nature is local. Our next Chapter 3 «*On some associated with groups numerical invariants*», is dedicated to those invariants – different ranks, and to the groups satisfying these or others conditions related to them.

Chapter 4 «*Normal subgroups and their influence on the group's structure*», is a survey of some most important classes of groups with different families of normal subgroups.

The last Chapter 5 «*On interposition of subgroups and its influence on their properties*», we consider a variety of generalized normal subgroups and their antipodes. Pronormal, abnormal, contranormal, transitively normal, nearly pronormal and other subgroups are in the spot there. We also focus on the arrangement of subgroups in a group, their interposition, and its influence on the group structure and subgroup properties.

The Contents of this book cannot absorbs all of the main topics of contemporary infinite group theory. Some important areas of research were not reflected. Perhaps, we will come back to this task, and create the continuation of the present book.

Authors.

Chapter 1

On some classes of generalized soluble groups

There are numbers of books, in which the properties of various classes of generalized soluble groups received complete and detailed consideration. Here, we are not going to discuss the entire variety of these classes; we will focus only on the generalized soluble groups that proved themselves to be the most popular and useful. The very definition of soluble groups based on the following concept of finite series of subgroups.

Recall that a group G is said to be ***soluble*** (or ***solvable***) if it has a finite chain of subgroups

$$\langle 1\rangle = G_0 \leq G_1 \leq \dots \leq G_n = G$$

such that G_{j-1} is normal in G_j and the factor – groups G_j/G_{j-1} are abelian for all $1 \leq j \leq n$.

The following concept of a commutator naturally arises and plays a significant role in soluble groups.

Let G be a group and x, y be elements of G. Recall that the product $x^{-1}y^{-1}xy = x^{-1}x^y$ is called the (simple) ***commutator of x and y*** and denotes by $[x,y]$. More generally, a ***commutator of weight n > 2*** is defined recursively by the rule

$$[x_1, \dots, x_{n-1}, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

The following brief notation is frequently used

$$[x,_n y] = [x_1, \dots, x_{n+1}], \text{ where } x_1 = x,\ x_2 = \dots = x_{n+1} = y.$$

Recall now some useful properties of commutators.

G.1. PROPOSITION. *Let G be a group, x, y, z ∈ G. Then:*

(i) $[x,y]^{-1} = [y,x]$;

(ii) $[xy,z] = [x,z]^y\,[y,z]$; $[x,yz] = [x,z]\,[x,y]^z$;

(iii) $[x,y^{-1}] = (y\,[x,y]\,y^{-1})^{-1}$; $[x^{-1},y] = (x\,[x,y]\,x^{-1})^{-1}$;

(iv) $[x,y^{-1},z]^y\,[y,z^{-1},x]^z\,[z,x^{-1},y]^x = 1$.

The notion of a commutator naturally extends to subsets. Let G be a group, X, Y be subsets of G. Then the ***commutator subgroup of X and Y*** is the subgroup, generated by all elements $[x,y]$, where $x \in X$, $y \in Y$. Note at ones that $[X,Y] = [Y,X]$.

More general, let $n > 2$ and $X_1, \dots, X_n$ be subsets of G, then put recursively

$$[X_1, \dots, X_{n-1}, X_n] = [[X_1, \dots, X_{n-1}], X_n].$$

As we already did it for elements, we will use the following brief notations:

$$[X,_n Y] = [X_1, \dots, X_{n+1}], \text{ where } X_1 = X, X_2 = \dots = X_{n+1} = Y.$$

If X, Y are subsets of a group G, then denote by X^Y the subgroup generated by all elements $x^y = y^{-1}xy$, where $x \in X$, $y \in Y$. If H is a subgroup of G, then X^H is a normal subgroup of $< X,H >$. Moreover, $X^H = X^{< X,H >}$. In particular, if X is a subgroup of H, then X^H is the minimal normal subgroup of H containing X. X^H is called the ***normal closure of X in H***.

Recall the following properties of normal closures.

G.2. PROPOSITION. *Let G be a group, X, Y be subsets of G, and H, K, L subgroups of G. Then:*

(i) $X^H = < X,[X,H] >$;

(ii) $[X,H]^H = [X,H]$;

(iii) if $K = < Y >$, *then* $[X,K] = [X,Y]^K$;

(iv) if $H = < X >$, $K = < Y >$, *then* $[H,K] = [X,Y]^{HK}$;

(v) if V is a normal subgroup of G containing any two subgroups of the following three subgroups $[H,K,L]$, $[K,L,H]$, $[L,H,K]$, *then V also contains the third subgroup.*

Let G be a group. Put $\delta_0(G) = G$, $\delta_1(G) = [G,G]$, and define recursively $\delta_{n+1}(G) = [\delta_n(G),\delta_n(G)]$ for all $n \in \mathbb{N}$. This series can be continued with the help of transfinite induction. Thus for every ordinal α we put $\delta_{\alpha+1}(G) = [\delta_\alpha(G),\delta_\alpha(G)]$, and $\delta_\lambda(G) = \bigcap_{\beta<\lambda} \delta_\beta(G)$ for all limit ordinals λ. Thus we construct the ***lower derived series***

$$G = \delta_0(G) \geq \delta_1(G) \geq \ldots \delta_\alpha(G) \geq \delta_{\alpha+1}(G) \geq \ldots \delta_\gamma(G) = D$$

of the group G. Every term of this series is a characteristic subgroup of G. The last term D of this series has the property $D = [D,D]$.

A group G is said to be ***perfect***, if $G = [G,G]$.

Let

$$\langle 1\rangle = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

be the series of subgroups such that G_{j-1} is normal in G_j and the factor – groups G_j/G_{j-1} are abelian, $1 \leq j \leq n$. Then $\delta_j(G) \leq G_{n-j}$, $0 \leq j \leq n$, in particular, $\delta_n(G) = \langle 1\rangle$. This remark shows that the lower derived series of a soluble group a series with the minimal possible length among all finite series having abelian factors. The length of the derived series of G is called the ***derived length of G*** and will be denoted by $\mathbf{dl}(G)$. This length is also called the class of solubility of the soluble group G. If $\mathbf{dl}(G) = 1$, then the group G is abelian, if $\mathbf{dl}(G) = 2$, then the group G is called ***metabelian***.

The commutator subgroups are also connected to another canonical series of a group G. Put $G = \gamma_1(G)$, $\gamma_2(G) = [G,G]$ and recursively $\gamma_{\alpha+1}(G) = [\gamma_\alpha(G),G]$ for all ordinals α and $\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G)$ for the limit ordinals λ. The series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \gamma_\alpha(G) \geq \gamma_{\alpha+1}(G) \geq \ldots \gamma_\delta(G)$$

is called the ***lower central series*** of the group G. The terms of this series are called ***hypocenters***. They are characteristic subgroups of G. The last term $\gamma_\delta(G)$ is called the ***lower hypocenter*** and we have $\gamma_\delta(G) = [\gamma_\delta(G),G]$.

If we consider the factor – group $G/\gamma_\alpha(G)$, then its center includes $\gamma_{\alpha+1}(G)/\gamma_\alpha(G)$. Let H and K be normal subgroups of G such that $H \le K$. Then the factor K/H is called ***G – central*** if $K/H \le \zeta(G/H)$ (or $C_G(K/H) = G$). Here $C_G(K/H)$ is the preimage in G of $C_{G/H}(K/H)$. If $C_G(K/H) \ne G$, then we say that the factor K/H is ***G – eccentric***.

Recall that a group G is said to be ***nilpotent*** if it has a finite chain of normal subgroups

$$\langle 1\rangle = G_0 \le G_1 \le \ldots \le G_n = G$$

such that every factor G_j/G_{j-1} is G – central, $1 \le j \le n$.

There is another canonical series, which is dual to lower central series and also connects to nilpotency. Put $\zeta_0(G) = \langle 1\rangle$, $\zeta_1(G) = \zeta(G)$ is the center of a group G, and recursively $\zeta_{\alpha+1}(G)/\zeta_\alpha(G) = \zeta(G/\zeta_\alpha(G))$ for all ordinal α, and $\zeta_\lambda(G) = \bigcup_{\beta<\lambda}\zeta_\beta(G)$ for all limit ordinals λ. The series

$$\langle 1\rangle = \zeta_0(G) \le \zeta_1(G) \le \ldots \zeta_\alpha(G) \le \zeta_{\alpha+1}(G) \le \ldots \zeta_\gamma(G)$$

is called the ***upper central series*** of the group G.

The terms of this series are called ***hypercenters***. They are characteristic subgroups of G. The last term $\zeta_\gamma(G)$ is called the ***upper hypercenter of G*** and we have $\zeta(G/\zeta_\gamma(G)) = \langle 1\rangle$. The notation $\zeta_\infty(G)$ is also frequently used for the upper hypercenter. The ordinal γ is called the ***central length*** of G and will be denoted by $\mathbf{zl}(G)$.

Now consider the basic properties of hypocenters and hypercenters.

G.3. PROPOSITION. *Let G be a group. Then:*

(i) if j, k are positive integers, then $[\gamma_j(G),\gamma_k(G)] \le \gamma_{j+k}(G)$;

(ii) if j, k are positive integers, then $\gamma_j(\gamma_k(G)) \le \gamma_{jk}(G)$;

(iii) if n, k are positive integers, then $\zeta_k(G/\zeta_n(G)) = \zeta_{k+n}(G)/\zeta_n(G)$;

(iv) if n, k are positive integers such that $k \ge n$, *then* $[\gamma_n(G),\zeta_k(G)] \le \zeta_{k-n}(G)$;

(v) if K is a normal subgroup of G, then $[\gamma_n(G),K] \le [K,_nG]$;

(vi) if K is a normal subgroup of G and H is a subgroup of G such that G = HK, then $\gamma_{n+1}(G) = \gamma_{n+1}(H)[K,_nG]$;

(vii) if H is a subgroup of G such that $G = H\zeta_n(G)$, *then* $\gamma_{n+1}(G) = \gamma_{n+1}(H)$.

G.4. PROPOSITION. *Let G be a group and A be an abelian normal subgroup of G. Then:*

(i) $[A,x] = \{ [a,x] \mid a \in A \}$ *for each element* $x \in G$;

(ii) $[A,\langle x \rangle] = [A,x]$ *for each element* $x \in G$;

(iii) if M is a subset of G, then $[A,\langle M \rangle]$ *is a product of all* $[A,x]$, *where* $x \in M$;

(iv) if M is a subset of G such that $G = \langle C_G(A),M \rangle$, *then* $[A,G]$ *is a product of all* $[A,x]$, *where* $x \in M$.

G.5. COROLLARY. *Let G be a group and M be a subset of G such that* $G = \langle [G,G],M \rangle$. *Then*

$$\gamma_n(G) = \langle \gamma_{n+1}(G), [x_1, \ldots, x_n] \mid x_1, \ldots, x_n \in M \rangle$$

for every positive integer n.

In particular, if the factor – group $G/[G,G]$ is finitely generated, then the factors $\gamma_n(G)/\gamma_{n+1}(G)$ are also finitely generated for all positive integer n.

Suppose that a group G has a finite central series, i.e. the series

$$\langle 1\rangle = G_0 \leq G_1 \leq \ldots \leq G_n = G$$

of normal subgroups such that $G_j/G_{j-1} \leq \zeta(G/G_{j-1})$, $1 \leq j \leq n$. Then $\gamma_j(G) \leq G_{n-j+1}(G)$ (so that $\gamma_{n+1}(G) = \langle 1\rangle$) and $G_j \leq \zeta_j(G)$ (so that $\zeta_n(G) = G$). This shows that the lower central series and the upper central series of a nilpotent group are the lowest by length of all finite central series of a group G. In particular, the lower central series and the upper central series have the same length that we call the ***nilpotency class of G*** and denote by **ncl**(G).

Using the upper and the lower central series, we obtain the following two natural generalizations of nilpotent groups.

A group G is said to be ***hypercentral*** *or* ***ZA – group*** if $\zeta_\infty(G) = G$.

A group G is said to be ***hypocentral*** *or* ***ZD – group*** if the lower hypocenter of G is trivial.

Despite of nilpotent groups, the classes of hypercentral and hypocentral groups have significant differences. Thus, the class of hypercentral groups inherits many good properties of nilpotent groups, while the class of hypocentral groups is very far from nilpotent groups. For example, by Magnus's theorem (see, for example, [**KAG1967**, Chapter IX, § 36]) every non – abelian free group has a lower central series of length ω, and hence belong to the class of hypocentral groups.

For introduction of some important classes of groups above, we used some concrete finite descending and ascending series of subgroups. Now we are on a position to consider the general case.

Let H be a subgroup of a group G and suppose that γ is an ordinal number. An ***ascending series*** from H to G is a set of subgroups

$$H = V_0 \leq V_1 \leq \ldots V_\alpha \leq V_{\alpha+1} \leq \ldots V_\gamma = G$$

such that V_α is normal in $V_{\alpha+1}$ and $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$ for all limit ordinals $\lambda < \gamma$.

In this case we call H an ***ascendant subgroup*** of G, unless the series involved is of finite length in which case, of course, H is said to be ***subnormal*** in G.

The subgroups V_α, $\alpha \leq \gamma$, are called the ***terms*** of this series and the factor – groups $V_{\alpha+1}/V_\alpha$ are called the ***sections*** of the series.

An ascending series is called ***normal*** if each term of this series is a normal subgroup of G. For normal series the factor – groups $V_{\alpha+1}/V_\alpha$ are called the ***factors*** of the series.

Let $\mathfrak{X}$ be a class of groups. This class can be defined also through the group theoretical property. A group G is called ***hyper – $\mathfrak{X}$ – group*** if it has an (in general, infinite) ascending series of normal subgroups terminating in G itself, whose factors belong to $\mathfrak{X}$. We will denote this class by $\mathbf{P}^{\uparrow}{}_{\mathbf{n}}\mathfrak{X}$. In particular, a group G is called ***hyperabelian*** if G has a normal ascending series, whose factors are abelian. This class of groups is one of the many generalizations of the class of soluble groups.

Further, we often shall meet the ***hyperfinite groups***, that are the groups having an ascending series of normal subgroups whose factors are finite.

The ascending series of normal subgroups

$$\langle 1\rangle = Z_0 \leq Z_1 \leq \ldots Z_\alpha \leq Z_{\alpha+1} \leq \ldots Z_\gamma = G$$

is called ***central*** if the corresponding factors $Z_{\alpha+1}/Z_\alpha$ are central, that is $Z_{\alpha+1}/Z_\alpha \leq \zeta(G/Z_\alpha)$ for all $\alpha < \gamma$.

We note the following characteristics and properties of hyperabelian and hypercentral groups.

G.6. PROPOSITION. *Let G be a group. Then:*

(i) G is hyperabelian if and only if every non – identity homomorphic image of G includes a non – identity normal abelian subgroup;

(ii) G is hypercentral if and only if every non – identity homomorphic image of G has a non – identity center.

G.7. COROLLARY. *Let G be a hyperabelian (respectively hypercentral) group. Then:*

(i) if H is a subgroup of G, then H is hyperabelian (respectively hypercentral) group;

(ii) if H is a normal subgroup of G, then the factor – group G/H is a hyperabelian (respectively hypercentral) group;

(iii) if H, K are the subgroups of G such that H is normal in K, then the section K/H is a hyperabelian (respectively hypercentral) group.

Dually we define the concept of descending series.

Let H be a subgroup of a group G and suppose that γ is an ordinal number. An ***descending series*** from G to H is a set of subgroups

$$G = V_0 \geq V_1 \geq \ldots V_\alpha \geq V_{\alpha+1} \geq \ldots V_\gamma = H$$

such that $V_{\alpha+1}$ is normal in V_α and $V_\lambda = \bigcap_{\beta<\lambda} V_\beta$ for all limit ordinals $\lambda < \gamma$.

In this case, we call H a ***descendant subgroup*** of G. When γ is finite, the subgroup H will be ***subnormal*** in G.

The subgroups V_α, $\alpha \leq \gamma$, are called the ***terms*** of this series and the factor – groups $V_\alpha/V_{\alpha+1}$ are called the ***sections*** of the series.

An descending series is called ***normal*** if each term of this series is a normal subgroup of G. For normal series the factor – groups $V_\alpha/V_{\alpha+1}$ are called the ***factors*** of the series.

Let $\mathfrak{X}$ is a class of groups. A group G is called ***hypo – $\mathfrak{X}$ – group*** if it has an (infinite in general) descending series of normal subgroups, terminating in <1>, whose factors belong to $\mathfrak{X}$. We will denote this class by $\mathbf{P}^{\Downarrow}{}_{\mathbf{n}}\mathfrak{X}$. In particular, a group G is called ***hypoabelian*** if G has a normal descending series, whose factors are abelian.

As we noted earlier, the hypercentral groups inherit many properties of nilpotent groups. It is well known that for each positive integer k there exists a nilpotent group G such that $\mathbf{ncl}(G) = k$. For the hypercentral groups we have the following analogy.

G.8. THEOREM (V.M. Glushkov [**GV1952[2]**]). *For every ordinal γ there exists a hypercentral group G such that* $\mathbf{zl}(G) = \gamma$.

The presence in a group G a finite central series entails the fact that $\gamma_k(G) = \langle 1 \rangle$ for some positive integer k. As the following simple example shows, for the groups having infinite central series this analogy is not retained.

Let P be a quasicyclic 2 – group, that is

$$P = \langle\, a_n \mid a_1^2 = 1,\ a_{n+1}^2 = a_n,\ n \in \mathbf{N} \,\rangle.$$

Being abelian, P has an automorphism α such that $\alpha(a) = a^{-1}$ for all $a \in P$. Clearly, $|\alpha| = 2$. This automorphism defines a semidirect product $G = P \leftthreetimes \langle d \rangle$ where $|d| = 2$ and $a^d = \alpha(a)$ for all $a \in P$. Then the series

$$\langle 1 \rangle < \langle a_1 \rangle < \langle a_2 \rangle < \ldots < \langle a_n \rangle < \ldots < P < G$$

is an upper central series of G. It is not hard to check that $P = [G,G]$, $\gamma_3(G) = [P,G] = P$. In particular, $\gamma_2(G) = \gamma_3(G) \neq \langle 1 \rangle$.

In this connection, we note the following specific result.

G.9. THEOREM (D.M. Smirnov [**SD1953**]). *Let G be a hypercentral group such that* $\mathbf{zl}(G) = \omega$ *(the first infinite ordinal). Then* $\gamma_{\omega+1}(G) = \langle 1 \rangle$.

Now note some other analogies of nilpotency.

G.10. THEOREM. *Let G be a hypercentral group. Then:*

(i) every subgroup of G is ascendant;

(ii) if H is a non – identity normal subgroup of G, then $H \cap \zeta(G) \neq \langle 1 \rangle$*;*

(iii) if A is a maximal normal abelian subgroup of G, then $A = C_G(A)$;

(iv) if A is a maximal normal abelian subgroup of G and there exist a positive integer k such that $A \leq \zeta_k(G)$, then G is nilpotent.

The following useful result due to S.N. Chernikov provides us with different types of characterizations of hypercentral groups.

G.11. THEOREM (S.N. Chernikov [**CSN1950**]). *Let G be a group.*

(i) G is hypercentral if and only if for each element $a \in G$ and every countable subset $\{ x_n \mid n \in \mathbb{N}\}$ of elements of G there exists a positive integer k such that

$$[\ldots[[a,x_1],x_2],\ldots,x_k] = 1.$$

(ii) G is hypercentral if and only if each of its countable subgroup is hypercentral.

We note that a class $\mathfrak{X}$ of groups is a ***countably recognizable*** class if the group $G \in \mathfrak{X}$ whenever every countable subset of elements of G lies in some $\mathfrak{X}$ – subgroup of G. Thus, **Theorem G.11** asserts that the class of hypercentral groups is a countably recognizable class and there is a similar result, due to R. Baer [**BR1962**], for the class of hyperabelian groups.

In a similar vein, if $\mathfrak{X}$ is a class of groups then a group G is called ***locally $\mathfrak{X}$ – group*** if every finite subset of G is contained in some $\mathfrak{X}$ – subgroup of G. The class of all locally $\mathfrak{X}$ – groups we denote by $\mathbf{L}\mathfrak{X}$. In particular, a group is called ***locally nilpotent***, ***locally soluble*** or ***locally finite*** in the cases when $\mathfrak{X}$ denotes respectively the class $\mathfrak{N}$ of all nilpotent groups, the class $\mathfrak{S}$ of all soluble groups, and the class $\mathfrak{F}$ of all finite groups.

As the following result shows, there is a very close relationship among the hypercentral and locally nilpotent groups.

G.12. THEOREM (A.I. Maltsev [**MAI1949**]). *Every hypercentral group is locally nilpotent.*

However, as can be seen from [**McL1959**], the classes of hyperabelian and locally soluble groups are distinct.

The locally nilpotent and hypercentral (respectively the locally soluble and hyperabelian) groups are parts of wider classes of groups which can be defined with the help of the following families of subgroups.

Let G be a group and $\mathfrak{S}$ be a family of subgroups of G. Then $\mathfrak{S}$ is said to be a ***Kurosh – Chernikov series*** if it satisfies the following conditions:

(KC 1) $\langle 1 \rangle, G \in \mathfrak{S}$;

(KC 2) for each pair A, B of the subgroups from $\mathfrak{S}$ either is $A \leq B$ or $B \leq A$;

(KC 3) for every subfamily $\mathfrak{L}$ of $\mathfrak{S}$ the intersection of all member of $\mathfrak{L}$ belongs to $\mathfrak{S}$ and the union of all member of $\mathfrak{L}$ belongs to $\mathfrak{S}$; in particular, for each non – identity element $x \in G$ the union V_x of all members of $\mathfrak{S}$ not containing x belongs to $\mathfrak{S}$, and the intersection Λ_x of all members of $\mathfrak{S}$ containing x belongs to $\mathfrak{S}$;

(KC 4) for each non – identity element $x \in G$ the subgroup V_x is normal in Λ_x.

The factor – groups Λ_x/V_x are called the ***sections of the series*** $\mathfrak{S}$.

If every subgroup of $\mathfrak{S}$ is normal in G, then $\mathfrak{S}$ is called a ***normal Kurosh – Chernikov series***. For this case the factor – groups Λ_x/V_x are called ***factors*** of the series $\mathfrak{S}$.

Such families were introduced by A.G. Kurosh and S.N. Chernikov in their fundamental classical article [**KuCh1947**]. In the paper [**KuCh1947**], such families were called by normal and invariant systems (families).

Let $\mathfrak{S}$ and $\mathfrak{S}_1$ are Kurosh – Chernikov series (respectively normal Kurosh – Chernikov series). If $\mathfrak{S} \subseteq \mathfrak{S}_1$, then we say that $\mathfrak{S}_1$ is a ***refinement*** of $\mathfrak{S}$. The refinement $\mathfrak{S}_1$ of $\mathfrak{S}$ is called ***proper***, if $\mathfrak{S}_1 \neq \mathfrak{S}$.

The Kurosh – Chernikov series (respectively normal Kurosh – Chernikov series) $\mathfrak{S}$ is called ***composition*** (respectively ***chief***) if $\mathfrak{S}$ has no a proper refinement.

A normal Kurosh – Chernikov series $\mathfrak{S}$ is called ***central*** if $\Lambda_x/V_x \leq \zeta(G/V_x)$ for each element x of a group G.

In this connections, if H, K be normal subgroups of G such that $H \leq K$, then the factor K/H is called a ***chief factor of G or G – chief*** if for every normal in G subgroup V such that $H \leq V \leq K$ is either $V = H$ or $V = K$. If $H = \langle 1 \rangle$, then we call K a ***minimal normal subgroup*** of G. It is a consequence of Zorn's Lemma that every group has chief factors although not every group possesses minimal normal subgroups.

We list now some of the Kurosh – Chernikov classes which were introduced in [**KuCh1947**].

A group G is called a ***SN – group***, if G has a Kurosh – Chernikov series, all of whose sections are abelian.

A group G is called a ***SI – group***, if G has a normal Kurosh – Chernikov series, all of whose factors are abelian.

A group G is called a $\overline{\boldsymbol{SN}}$ – ***group***, if all of sections of every composition Kurosh – Chernikov series of G are abelian.

A group G is called a $\overline{\boldsymbol{SI}}$ – ***group***, if all of factors of every chief Kurosh – Chernikov series of G are abelian.

A group G is called a ***Z – group***, if G has a central Kurosh – Chernikov series.

A group G is called a $\overline{\boldsymbol{Z}}$ – ***group***, if every chief Kurosh – Chernikov series of G is central.

A group G is called an ***Ñ – group*** if for every subgroup H of G there is a Kurosh – Chernikov series $\mathfrak{S}$ such that H $\in \mathfrak{S}$.

In the books [**KM1982**, **KAG1967**, **RD1972**], the connections and relationships of these classes were discussed in detail. That is why we are not going to consider it here. First of all, observe that these properties local. In the article [**KuCh1947**], one can find a direct group – theoretical proof of this fact. However, for the same result, one can use the developed by A.I. Malcev in [**MAI1941**] methods of proving of local theorems. Such proof one can find in the book [**KM1982**, § 22].

G.13. THEOREM (A.G. Kurosh, S.N. Chernikov [**KuCh1947**]). *If every finitely generated subgroup of a group G is a SN – group (respectively a SI – group,* $\overline{SN}$ *– group,* $\overline{SI}$ *– group, Z – group, Ñ – group), then the group G itself is a SN – group (respectively a SI – group,* $\overline{SN}$ *– group,* $\overline{SI}$ *– group, Z – group, Ñ – group).*

It worthy to mentioned that the local theorem for Ñ – group has also been proven in the paper of R. Baer [**BR1940**]. Point on some important corollaries of these local theorems.

G.14. COROLLARY (A.G. Kurosh, S.N. Chernikov [**KuCh1947**]). *Let G be a locally soluble group. Then:*

(i) G has a normal Kurosh – Chernikov series, whose factors are abelian;

(ii) every chief factor of G is abelian; in particular, a simple locally soluble group has prime order.

G.15. COROLLARY (A.G. Kurosh, S.N. Chernikov [**KuCh1947**]). *Let G be a locally nilpotent group. Then:*

(i) G has a central Kurosh – Chernikov series;

(ii) every chief factor of G is central and hence has a prime order.

G.16. COROLLARY (A.G. Kurosh, S.N. Chernikov [**KuCh1947**]). *Let G be a group. Then:*

(i) G is an Ñ – group if and only if G satisfies the following condition: if M, L are subgroups of G such that M is maximal in L, then M is normal in L;

(ii) every locally nilpotent group is an Ñ – group; in particular, every maximal subgroup of a locally nilpotent group is normal, and for every subgroup H of G there is a Kurosh – Chernikov series $\mathfrak{S}$ such that $H \in \mathfrak{S}$.

The following groups form an important specific case of Ñ –groups.

A group G is called an ***N – group*** if every subgroup of G is ascendant.

For N – groups we have

G.17. PROPOSITION. *Let G be a group. Then:*

(i) G is an N – group if and only if $\mathbf{N}_G(H) \neq H$ for every proper subgroup H (A.G. Kurosh, S.N. Chernikov [**KuCh1947**]);

(ii) every N – group is locally nilpotent (B.I. Plotkin [**PB1951**]).

Consider in detail some properties of locally nilpotent groups.

Let G be a group and π be a set of primes. Denote by $\mathbf{O}_\pi(G)$ the largest normal π – subgroup of the group G. If $\pi = \Pi(G)$, then $\mathbf{O}_\pi(G)$ is the largest normal periodic (torsion) subgroup of G. We denote the subgroup $\mathbf{O}_{\Pi(G)}(G)$ by $\mathbf{Tor}(G)$. The subgroup $\mathbf{Tor}(G)$ is called the ***periodic part*** of the group G.

Now we show the following important properties of locally nilpotent groups.

G.18. PROPOSITION. *Let G be a locally nilpotent group. Then:*

(i) for every set π of primes $\mathbf{O}_\pi(G)$ coincides with the subset of all π – elements of G;

(ii) the set of all elements of G, having finite order, coincides with ***Tor***(G)*, and G/* ***Tor***(G) *is torsion – free;*

(iii) for each prime p, the set ***Tor***$_p(G)$ *of elements of p – power order is a characteristic subgroup of* ***Tor***(G) *and* ***Tor***$(G) = \mathbf{Dr}_{p \in \Pi(G)}$ ***Tor***$_p(G)$.

The following properties of torsion – free locally nilpotent groups are essential and useful.

G.19. PROPOSITION (A.I. Maltsev [**MAI1949**]). *Let G be a torsion –free locally nilpotent group. Then:*

(i) if x, y are elements of G such that $x^n = y^n$ for some positive integer n, then $x = y$;

(ii) if x, y are elements of G such that $y^t x^k = x^k y^t$ for some positive integers k, t, then $yx = xy$.

Let G be a group. Recall that a subgroup H of a group G is called ***pure*** in G if either $\langle g \rangle \cap H = \langle g \rangle$ or $\langle g \rangle \cap H = \langle 1 \rangle$ for each element $g \in G$.

As the following theorem shows, locally nilpotent torsion – free groups have quite many pure subgroups.

G.20. THEOREM (A.I. Maltsev [**MAI1949**]). *Let G be a torsion – free locally nilpotent group. Then:*

(i) if M is a subset of G, then $\mathbf{C}_G(M)$ *is a pure subgroup of G;*

(ii) if G is a torsion – free group and A is a normal subgroup of G, then A is pure if and only if G/A is torsion – free;

(iii) the intersection of every family of pure subgroups is a pure subgroup;

(iv) every term of the upper central series of G is pure; in particular, if G is hypercentral, then all the factors of the upper central series are torsion – free.

We can complement these properties with the following property which has been obtained by V.M. Glushkov.

G.21. THEOREM (V.M. Glushkov [**GV1952[1]**]). *Let G be a torsion –locally nilpotent group. Then G has a Kurosh – Chernikov series, whose terms are pure in G.*

Consider now the important class of hyper – (locally nilpotent) groups, which B.I Plotkin named by radical groups. The base results justifying the value of introduction of such groups were the following useful theorems.

G.22. THEOREM (B.I. Plotkin [**PB1955**], K.A. Hirsch [**HKA1955**]). *Let G be a group. Then the subgroup generated by all normal locally nilpotent subgroups is a normal locally nilpotent subgroup of G.*

G.23. PROPOSITION (B.I. Plotkin [**PB1955**]). *Let G be a group. Then:*

(i) the subgroup, generated by all normal periodic π–subgroups (π is some set of primes) of G is a periodic normal π–subgroup of G;

(ii) if H is an ascendant periodic π – subgroup of G, then H^G is a normal periodic π – subgroup of G;

(iii) if H is an ascendant locally nilpotent subgroup of G, then H^G is a normal locally nilpotent subgroup of G.

These results show that every group G contains a unique maximal normal locally nilpotent subgroup that is called the ***locally nilpotent radical*** or ***the Hirsch – Plotkin radical*** of G. This subgroup will be denoted by **Ln**(G). Of course, **Ln**(G) is a characteristic subgroup of G. **Proposition G.23** shows that if G includes a non – identity ascendant locally nilpotent subgroup, then its locally nilpotent radical is non – identity.

A group G is called ***radical*** if it is hyper – (locally nilpotent). Our remarks above show that the class of radical groups is precisely the class of groups with an ascending series each factor of which is locally nilpotent.

We define the ***radical series***

$$\langle 1\rangle = R_0 \le R_1 \le \ldots R_\alpha \le R_{\alpha+1} \le \ldots R_\gamma$$

of the group G by:

$R_1 = \mathbf{Ln}(G)$,

$R_{\alpha+1}/R_\alpha = \mathbf{Ln}(G/R_\alpha)$ for all ordinals $\alpha < \gamma$ and

$R_\lambda = \bigcup_{\beta<\lambda} R_\beta$ for limit ordinals $\lambda < \gamma$,

$\mathbf{Ln}(G/R_\gamma) = \langle 1\rangle$.

It may happens that **Ln**(G) is identity. The group G is radical if and only if R_γ of this series coincides with G.

List some useful properties of radical groups.

G.24. PROPOSITION. *Let G be a radical group. Then:*

(i) if H is a subgroup of G, then H is a radical group;

(ii) if H is a normal subgroup of G, then the factor – group G/H is a radical group;

(iii) if H, K are subgroups of G such that H is normal in K, then the section K/H is a radical group;

(iv) $\mathbf{C}_G(\mathbf{Ln}(G)) \leq \mathbf{Ln}(G)$.

Consider another useful concept of the radical of a group. The following statement could be used for a proof of its existence, while it frequently implements in different situations.

G.25. PROPOSITION. *Let G be a finitely generated group and C be a subgroup of G. If C has finite index, then C is finitely generated.*

G.26. COROLLARY. *Let G be a group and suppose that G includes a normal locally finite subgroup H such that G/H is also locally finite. Then G is locally finite.*

G.27. PROPOSITION (B.I. Plotkin [**PB1955**]). *Let G be a group. Then:*

(i) the subgroup generated by the normal locally finite subgroups of G is a normal locally finite subgroup of G;

(ii) if H is an ascendant locally finite subgroup of G, then H^G is a normal locally finite subgroup of G;

(iii) the subgroup, generated by the normal radical subgroups of G is a normal radical subgroup of G;

(iv) if H is an ascendant radical subgroup of G, then H^G is a normal radical subgroup of G.

It is also easy to see that a product of normal soluble subgroups is locally soluble. However

G.28. THEOREM (Ph. Hall [**RD1972**, Theorem 8.19.1).

(i) There exists a finitely generated non – hyperabelian group, which is the product of two normal locally soluble hyperabelian subgroups.

(ii) There exists a finitely generated group, which is the product of two normal locally soluble subgroups, but which is not an SI – group or radical group.

Proposition G.27 shows that if G includes a non – identity ascendant locally finite subgroup, then it has the largest non – identity locally finite normal subgroup (***the locally finite radical***) **Lf**(G). Of course, **Lf**(G) is a characteristic subgroup of G.

A group G is called ***generalized radical***, if G has an ascending series whose factors are locally nilpotent or locally finite.

It easily follows from the definition that a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the former case, the locally nilpotent radical of G is non – identity. In the latter case, by

Proposition G.27, G contains a non – identity normal locally finite subgroup, so the locally finite radical is non – identity. Thus, every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors.

As always, we list of the main properties of generalized radical groups.

G.29. PROPOSITION. *Let G be a group. Then:*

(i) G is a generalized radical if and only if every non – identity homomorphic image of G includes a non – identity ascendant locally nilpotent or locally finite subgroup;

(ii) if G is a generalized radical group and H is a subgroup of G, then H is generalized radical;

(iii) if G is a generalized radical group and H is a normal subgroup of G, then the factor – group G/H is generalized radical;

(iv) if G is a generalized radical group and H, K are subgroups of G such that H is normal in K, then the section K/H is a generalized radical group;

(v) if G is a finitely generated periodic generalized radical group, then G is finite. Consequently, a periodic locally generalized radical group is locally finite.

Let G be a group and $\mathfrak{X}$ be a class of groups and

$$\mathbf{Res}_{\mathfrak{X}}(G) = \{ H \mid H \text{ is a normal subgroup such that } G/H \in \mathfrak{X} \}.$$

Then the intersection $G^{\mathfrak{X}}$ of all normal subgroups from the family $\mathbf{Res}_{\mathfrak{X}}(G)$ is called the $\mathfrak{X}$ ***– residual*** of a group G. If $\mathbf{Res}_{\mathfrak{X}}(G)$ has the least element L, then $L = G^{\mathfrak{X}}$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$. However, in general, $G/G^{\mathfrak{X}} \notin \mathfrak{X}$.

If $\mathfrak{X} = \mathfrak{A}$ is the class of all abelian groups, then the $\mathfrak{A}$ – residual $G^{\mathfrak{A}}$ is exactly the derived subgroup [G,G] of a group G. In particular, $G/G^{\mathfrak{A}} \in \mathfrak{A}$.

If $\mathfrak{X} = \mathfrak{N}_{\mathbf{c}}$ is the class of all nilpotent groups, having nilpotency class at most **c**, then the $\mathfrak{N}_{\mathbf{c}}$ – residual $G^{\mathfrak{N}\mathbf{c}}$ is exactly the subgroup $\gamma_{\mathbf{c}+\mathbf{1}}(G)$. In particular, $G/G^{\mathfrak{N}\mathbf{c}} \in \mathfrak{N}_{\mathbf{c}}$.

But if $\mathfrak{X} = \mathfrak{N}$ is the class of all nilpotent groups, then in general $G/G^{\mathfrak{N}} \notin \mathfrak{N}$. Moreover, this factor – group can be not locally nilpotent. For example, as we remark above, if G is a free group, then by Magnus's theorem $\gamma_{\omega}(G) = <1>$. It follows that $G^{\mathfrak{N}} = <1>$, so that $G/G^{\mathfrak{N}}$ is a free group.

Let $\mathfrak{X}$ be a class of groups. A group G is called ***residually*** ***$\mathfrak{X}$ – group*** if for each non – identity element $g \in G$ there is a normal subgroup H_g such that $g \notin H_g$ and $G/H_g \in \mathfrak{X}$.

G.30. PROPOSITION. *Let G be a group and $\mathfrak{X}$ be a class of groups. Then G is a residually $\mathfrak{X}$ – group if and only if* $\cap\ \mathbf{Res}_{\mathfrak{X}}(G) = <1>$.

We will denote the class of the residually $\mathfrak{X}$ – groups by $\mathbf{R}\mathfrak{X}$. If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite group, then we obtain the familiar class $\mathbf{R}\mathfrak{F}$ of all ***residually finite*** groups. If p is a prime, we let $\mathfrak{F}_p$ denote the class of all finite p – groups and in this way we obtain the class $\mathbf{R}\mathfrak{F}_p$ of the ***residually $\mathfrak{F}_p$ – groups***. It is well – known that every free group is a residually $\mathfrak{F}_p$ – group for each prime p. This result was originally obtained by K. Iwasawa [**IK1943**].

It is easy to see that $G^{\mathfrak{X}}$ is always a characteristic subgroup of G and the factor – group $G/G^{\mathfrak{X}}$ is always a residually $\mathfrak{X}$ – group.

If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups then $G^{\mathfrak{X}} = G^{\mathfrak{F}}$ is the ***finite residual*** of a group G.

We will need also the following concept and related results.

Suppose that

$$\langle 1 \rangle = G_0 \lhd G_1 \lhd \ldots \lhd G_j \lhd G_{j+1} \lhd \ldots \lhd G_n = G,$$

is a finite chain of subgroups of a group G. An automorphism α of G is said to ***stabilize*** this chain if $\alpha(xG_j) = xG_j$ for every $x \in G_{j+1}$ and all j, $0 \leq j \leq n - 1$.

The set of all such automorphisms stabilizing the chain in the above definition is a subgroup of **Aut** (G) called the ***stability group*** of the chain.

G.31. PROPOSITION (L.A. Kaloujnine [**K1953**]). *Let G be a group. Let G be a group having a finite series*

$$\langle 1 \rangle = G_0 \leq G_1 \leq \ldots \leq G_{n-1} \leq G_n = G$$

of normal subgroups. Then the stability group of this series is nilpotent of class at most n – 1.

This result has been improved considerably by Ph. Hall in [**HP1958**] where it was shown that the stability group of any chain of subgroups of length n (without a normality assumption) is nilpotent of class at most ½ n(n – 1).

Chapter

2

On some finiteness conditions in infinite groups

It is understood that the study of groups is impossible without additional restrictions. Therefore, the question of the choice of such restrictions is one of the most important in group theory. Those restrictions should be natural and sufficiently well – founded. As it was proven historically, the restrictions based on the finiteness conditions completely satisfy this requirement.

Group theory emerged and initially began developing as a theory of groups of permutations. Then, having accumulated many important results, it goes to a new level by developing the abstract theory of finite groups. The progress of this theory was quite impressive. At the same time, various areas of mathematics, especially geometry, topology, the theory of automorphic functions, and others begin to deal with problems in which various infinite groups are necessary involved. That is why the establishment of the theory of infinite groups becomes a natural and vital. However it was unclear, how to proceed with that and in what direction to go? A good possible related role model was ring theory, which by that time was already a well – developed branch of algebra and where the separation by finite and infinite objects was not essential. The maximal and minimal conditions played central role in ring theory. Therefore it was natural to use in groups such approaches have proven to be productive and effective in ring and module theories.

It should be noted that in 1920 – 1930, the contacts between German and Soviet mathematicians were quite close. In particular, such famous mathematicians as O.Yu. Schmidt, the founder of the Russian and later Soviet group – theoretical school, and P.S. Alexandrov, a teacher of A.G. Kurosh, had extensive personal connections with many leading German mathematicians and often visited Germany. Many German mathematicians, in particular, one of the founders of modern algebra E. Noether had lectured at Moscow State University that time.

In infinite group theory, it was also a natural tendency to use quite diverse and rich experience accumulated in investigation of finite groups. This is largely determined the fact that one of the first areas of infinite group theory was dedicated to the groups with finiteness conditions (i.e. to the groups with the restrictions, which always take place for any finite group). The development of this area was strongly supported and stimulated by O.Yu. Schmidt, who led the famous seminar at Moscow State University, and by the second head of this seminar A.G. Kurosh. The formation and development of such a great new theory of infinite groups, which was called «*The groups with finiteness conditions*», was held in inseparable conjunction with the formation of the theory of generalized solvable groups. In particular, all of the classes of generalized soluble groups, which we introduced in Chapter 1, arose in the course of the study groups with finiteness conditions. This work was done with direct involvement of such well – known algebraists as A.I. Maltsev, V.M. Glushkov, V.S. Charin, B.I. Plotkin, R. Baer, K. Hirsch, B.H. Neumann, Ph. Hall, K. Gruenberg and others. Under *a finiteness condition* in group theory one refers to such property which inherited by all finite groups and at least one infinite group does not possess this property. The imposing of the finiteness conditions allows you to select such infinite groups that retain certain properties of finite groups. The finiteness conditions helped to select many natural classes of groups that are at the intersection of finite and infinite groups.

As noted above, the finiteness conditions, which began to be studied first, were the minimal and maximal conditions or the descending and ascending chain conditions. These concepts are related to the mutual arrangement of subgroups. Now we formulate these concepts in the most general form for ordered sets.

Let M be an ordered set with the ordering $\leq$. Recall that M satisfies ***the minimal*** (or ***the minimum***) ***condition*** if every non – empty subset of M has a minimal element.

We say that M satisfies ***the descending chain condition*** if for every descending chain

$$a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots$$

of elements of M, there is some $k \in \mathbf{N}$ such that $a_k = a_{k+1} = = a_{k+2} = \ldots$.

In this regard, we have the following well – known result.

F.1. PROPOSITION. *An ordered set M satisfies the minimal condition if and only if M satisfies the descending chain condition.*

We also have the dually situation.

An ordered set M satisfies ***the maximal*** (or ***the maximum***) ***condition*** if every non – empty subset of M has a maximal element.

We say that M satisfies ***the ascending chain condition*** if for every ascending chain

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$$

of elements of M there is some $k \in \mathbf{N}$ such that $a_k = a_{k+1} = = a_{k+2} = \dots$.

F.1A. PROPOSITION. *An ordered set M satisfies the maximal condition if and only if M satisfies the ascending chain condition.*

Note also the following general property.

F.2. PROPOSITION. *Let M be an ordered set. Then M satisfies the minimal (respectively the maximal) condition if and only if every its linearly ordered subset is well – ordered by descending (respectively by ascending).*

A subgroup theoretical property $\mathfrak{P}$ is a property that certain subgroups of a group G posses such that whenever a subgroup H of a group G has the property $\mathfrak{P}$, then θ(H) also has property $\mathfrak{P}$ for each isomorphism θ of G with some other group. Typical examples of $\mathfrak{P}$ are the properties of being normal, subnormal, abelian, subnormal abelian, central and so on.

Let $\mathfrak{P}$ be a subgroup theoretical property. The group G is said to ***satisfy the minimum*** (or ***the minimal***) ***condition on*** ***$\mathfrak{P}$ – subgroups*** (**Min** – $\mathfrak{P}$ for short) if the ordered by inclusion set of all $\mathfrak{P}$ – subgroup of G satisfies the minimal condition.

Dually, a group G is said to ***satisfy the maximal condition on $\mathfrak{P}$ – subgroups*** (**Max** – $\mathfrak{P}$ for short) if the ordered by inclusion set of all $\mathfrak{P}$ – subgroup of G satisfies the maximal condition.

For example, if $\mathfrak{P}$ is the property of being a subgroup, then the condition **Min** – $\mathfrak{P}$ is called the ***minimal condition***, and is often abbreviated to **Min**. If $\mathfrak{P}$ represents the property of being an abelian subgroup, then we obtain the condition **Min – ab**, the minimal condition on abelian subgroups. Conditions **Min – p** and **Min – sn** denote respectively the cases when $\mathcal{P}$ represents the property of being a p – subgroup (for the prime p) and subnormal subgroup respectively.

The minimal condition has been playing an important role both in ring theory and group theory for a number of years. However, the structure of groups with **Min** is not as well understood as the corresponding structure of rings. Since groups with the minimal condition are highly relevant to the topic in which we are interested, we shall spend some time obtaining some of results concerning the structure of groups with **Min**. Of course, all finite groups have **Min**.

The following proposition lists some of the properties of groups with the minimal condition.

F.3. PROPOSITION. *Let G be a group. Then:*

*(i) if G satisfies **Min**, then every subgroup H of G satisfies **Min**;*

*(ii) if G satisfies **Min** and L is a normal subgroup of G, then the factor – group G/L satisfies **Min**;*

*(iii) if L is a normal subgroup of G such that L and G/L satisfy **Min**, then G satisfies **Min**;*

(iv) suppose that a group G has a finite series of subgroups

$$\langle 1 \rangle = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

*every factor H_j/H_{j-1} of which satisfies **Min**, $1 \le j \le n$, then G satisfies **Min**;*

*(v) if $G = H_1 \times H_2 \times \ldots \times H_n$, where each group H_j satisfies **Min**, $1 \le j \le n$, then G satisfies **Min**;*

*(vi) let G be a group and H a subgroup of G such that $H = \mathbf{Dr}_{\lambda \in \Lambda} H_\lambda$; if G satisfies **Min**, then the set Λ is finite;*

(vii) the group G satisfies the minimal condition if and only if every countable subgroup of G satisfies the minimal condition.

The most important examples of infinite groups with the minimal condition are the ***Prüfer groups of type p***$^\infty$. Initially such groups were called the ***quasicyclic p – groups***, but this latter terminology is now often used for groups whose proper subgroups are cyclic. Among such groups, there are some that are very different from the Prüfer groups. We now describe the structure of the Prüfer p – groups.

Let p be a prime. For each positive integer number $n \geq 1$, let $G_n = \langle a_n \rangle$ be a cyclic group of order p^n, and for each $n \in \mathbb{N}$, $\theta_n: G_n \to G_{n+1}$ be the monomorphism defined by the rule:

$$\theta(a_n) = a_{n+1}^p,\ n \in \mathbb{N}.$$

In this way, we can think of G_n as a subgroup of G_{n+1} and hence we can form the group

$$\mathbf{C}_{p^\infty} = \bigcup_{n \in \mathbb{N}} G_n,$$

which is a union of a chain of cyclic p – groups of orders p, p^2, … .

Let p be a prime. The obtained above group $\mathbf{C}_{p^\infty}$ is called a ***Prüfer group of type p***$^\infty$***.***

There are numerous useful descriptions of this group. It can be realized in terms of generators and relations as

$$\mathbf{C}_{p^\infty} = \langle a_n \mid a_1^p = 1,\ a_{n+1}^p = a_n,\ n \in \mathbb{N} \rangle.$$

A Prüfer p – group also arises somewhat more concretely. It can be thought of as the multiplicative group of complex p^{th} roots of unity, or as the set of elements of p – power order in the additive abelian group $\mathbb{Q}/\mathbb{Z}$, thus $\mathbb{C}_{p^{\infty}} = \mathbf{Tor}_p(\mathbb{Q}/\mathbb{Z})$. The Prüfer groups play a very important role in group theory. They are the main examples of infinite groups in which every proper subgroup is finite. Such groups are called ***quasifinite.***

Note that $\mathbb{C}_{p^{\infty}}$ satisfies the minimal condition on subgroups. Thus, group $\mathbb{C}_{p^{\infty}}$ is a fundamental example of a group satisfying the minimal condition on subgroups.

A Prüfer p – group has another interesting property.

A group G is called ***divisible*** (many authors use this word only if the group is abelian, and reserve the term ***radicable*** in the general case), if for each element $g \in G$ and each $n \in Z$, the equation $x^n = g$ always has a solution in a group G.

Put $G^n = \langle g^n \mid g \in G \rangle$. Note that if G is an abelian group, then $G^n = \{ g^n \mid g \in G \}$. Clearly, an abelian group G is divisible if and only if $G = G^n$ for each $n \in \mathbb{N}$.

Another example of a divisible abelian group is the additive group $\mathbb{Q}$ of rational numbers. These two examples of divisible abelian groups are central. As the following theorem shows, any divisible abelian group could be constructed using these two examples.

F.4. THEOREM. *Let G be an abelian divisible group. Then* $G = (\mathbf{Dr}_{\lambda \in \Lambda} D_\lambda) \times (\mathbf{Dr}_{p \in \Pi(G)} T_p)$, *where* $T_p = \mathbf{Dr}_{\mu \in M(p)} Q_\mu$ *and* $D_\lambda \cong \mathbb{Q}$ *for all* $\lambda \in \Lambda$, $Q_\mu \cong \mathbb{C}_{p^{\infty}}$ *for all* $\mu \in M(p)$ *(M(p) is a set of indexes which depends on p). The cardinal numbers* $|\Lambda|, |M(p)|$ *are invariants of the group G.*

The following important property of abelian divisible groups should be noted.

F.5. THEOREM (R. Baer [**BR1940[1]**). *Let G be an abelian group. If D is a divisible subgroup of G, then $G = D \times C$ for some subgroup C. Moreover, for every subgroup B such that $B \cap D = <1>$ there exists a subgroup E such that $B \leq E$ and $G = D \times E$.*

It follows that every abelian group G has a maximal divisible subgroup D = **Div**(G), which is called the ***divisible part*** of G, and G = D × R for some subgroup R, which contains no non – identity divisible subgroups. Such a subgroup R is said to be ***reduced***.

The next concept is a generalization of a concept of divisibility. A group G is called $\mathfrak{F}$ ***– perfect*** if G does not include proper subgroups of finite index.

Clearly, every divisible abelian group is $\mathfrak{F}$ – perfect, and conversely, every abelian $\mathfrak{F}$ – perfect group is divisible. This result can easily be proven with the help of the following important theorem from abelian group theory. It will also be useful farther.

F.6. THEOREM (H. Prüfer [**PH1923**]). *Let G be an abelian group. If there exists a positive integer n such that $G^n = <1>$, then G is a direct product of cyclic subgroups.*

Suppose now that $G \neq G^n$ for some positive integer n, then the orders of elements of the factor – group G/G^n divide n. By

Theorem F6, G/G^n is a direct product of finite cyclic subgroups. In particular, it has a proper subgroup of finite index. This contradiction shows that $G = G^n$ for each $n \in \mathbf{N}$, so that G is divisible.

By considering a minimal subgroup of finite index in a group with the minimal condition, it is possible to deduce the following result.

F.7. PROPOSITION. *Every group satisfying the minimal condition on subgroups has a normal* $\mathfrak{F}$ *– perfect subgroup of finite index.*

The structure of abelian groups with the minimal condition has been obtained quite a long time ago.

F.8. THEOREM (A.G. Kurosh [**KAG1932**]). *Let G be an abelian group. Then G has the minimal condition if and only if G is a direct product of finitely many Prüfer groups and finite group.*

A group G is called a ***Chernikov group***, if G includes a normal subgroup D of finite index, which is a direct product of finitely many Prüfer subgroups.

Such groups were named in honor of S.N. Chernikov, who made an extensive study of groups with the minimum condition. S.N. Chernikov called these groups ***extremal***. He played a key role in the study of non – abelian satisfying the minimal conditions for all subgroups. S.N. Chernikov has formulated the famous problem of whether every group satisfying the minimum condition is extremal, which has long been one of the impetuses for the development of this area.

The subgroup D is easily seen to be a maximal divisible subgroup of G, so that D is the divisible part of G. Thus, by **Proposition F.3**, every Chernikov group satisfies the minimal condition. Note the following properties of Chernikov groups.

F.9. PROPOSITION. *Let G be a group. Then:*

(i) if G is a Chernikov group, then every subgroup H of G is also Chernikov;

(ii) if G is a Chernikov group and L is a normal subgroup of G, then the factor – group G/L is Chernikov;

(iii) if L is a normal subgroup of G such that L and G/L are divisible Chernikov groups, then G is divisible, and hence abelian, Chernikov group;

(iv) if L is a normal subgroup of G such that L and G/L are Chernikov groups, then G is a Chernikov group;

(v) suppose that a group G has a finite series of subgroups

$$\langle 1\rangle = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G,$$

every factor H_j/H_{j-1} of which is a Chernikov group, $1 \le j \le n$; then G is a Chernikov group.

The study of groups with minimal condition was carried out step by step. In this passing, some new classes of infinite groups, which are now perceived by the ordinary and usual, were discovered and described. We will not show here all of these steps. We present only basic results.

F.10. THEOREM (S.N. Chernikov [**CSN1940[1]**], [**CSN1940[2]**]). *Let G be a locally soluble group. Then G has the minimal condition if and only if G is the Chernikov group.*

The famous problem on the existence of infinite groups whose all proper subgroups are finite has been formulated by O. Yu. Schmidt in connection of study of Chernikov groups. The first known examples of such groups were Prüfer groups of type p^{∞}. As a result of a quite long process, the following impressive theorem was obtained.

F.11. THEOREM (M.I. Kargapolov [**KMI1963**], Ph. Hall, C.R. Kulatilaka [**HK1964**]). *Let G be an infinite locally finite group. Then G includes an infinite abelian subgroup. In particular, if every proper subgroup of G is finite, then G is abelian, and therefore is a Prüfer p – group for some prime p.*

This result was a stimulus for the description of locally finite groups satisfying the minimal condition for all subgroups, and soon it his description was obtained.

F.12. THEOREM (O.H. Kegel, B.A.F. Wehrfritz [**KW1970**], V.P. Shunkov [**ShV1970**]). *Let G be a locally finite group. Then G has the minimal condition if and only if G is the Chernikov group.*

As the following obtained by O.Yu. Schmidt result shows, a special place here is occupied by 2 – groups.

F.13. THEOREM (O.Yu. Schmidt [**SchO1959**]). *Let G be a 2 – group. If G satisfies the minimal condition, then G is locally finite, and hence is the Chernikov group.*

Note that the solution of the problems of O.Yu. Schmidt and S.N. Chernikov was obtained outside the theory of groups with finiteness conditions with the help of a completely different technique.

F.14. THEOREM (A.Yu. Olshanskii [**OA1980**]). *There exists an infinite group G such that every proper subgroup of G has prime order. Moreover, if x and y are elements of G such that* $|x| = |y|$, *then the subgroups* $< x >$ *and* $< y >$ *are conjugate.*

F.15. THEOREM (A.Yu. Olshanskii [**OA1982**]). *Let p be a prime such that* $p > 10^{75}$. *Then there exists an infinite group G such that every proper subgroup of G has order p.*

The group from **Theorem F.15** is called the ***Tarski monster***.

F.16. THEOREM (V.N. Obraztsov [**OVN1989**]). *There exists an uncountable group satisfying the minimal condition for all subgroups.*

The examples mentioned in **Theorems F14 – F16**, along with other unique examples, held a kind of watershed in the theory of infinite groups. Thus, a vast well – developed theory with its own methodology and tools was developed. It looks like a well – crafted and well – groomed valley that surrounds a large and dense forest. In the forest, in different directions, there are glades, wide or narrow, but not so much to find out what lurks in the forest. As S.N. Chernikov noted in this regard, these examples «underline the completion of the positive results» and lead to a clearer understanding of an important fact that the theory of generalized solvable groups and the theory of groups with finiteness conditions (similar to other traditional areas of group theory, such as the theory of finite groups, abelian group theory linear groups, etc.), is an important part of a general theory of specific groups that have their own research goals and approaches, its own methodology and rich history.

We introduce now the following concept, which we will very often use. Let $\mathfrak{X}$ be a class of groups. Recall that a group G is called ***almost*** $\mathfrak{X}$ ***– group*** if G includes a normal subgroup H $\in \mathfrak{X}$ such that the index |G:H| is finite. The class of all almost $\mathfrak{X}$ – group is denoted by $\mathfrak{X}\mathfrak{F}$. In particular, a group is ***almost soluble*** if it has a soluble subgroup of finite index.

For groups of the class $\mathfrak{X}\mathfrak{F}$ one uses another term, namely, a ***virtually*** $\mathfrak{X}$ ***– group.*** In our opinion, it does not reflect properly the essence of the notion, so we will use the term an ***almost*** $\mathfrak{X}$ ***– group***.

The case of the maximal condition on subgroups was much less complicated. R. Baer has considered almost soluble groups satisfying **Max** in the paper [**BR1956**]. The result of R. Baer can be easily expanded to the next theorem.

F.17. THEOREM. *Let G be a locally generalized radical group. Then G satisfies the maximal condition for all subgroups if and only if G is polycyclic – by – finite.*

As for the groups with **Min**, for the groups satisfying **Max** the following question is naturally raised: is any group with the maximum condition for all subgroups almost polycyclic? (a problem of R. Baer). A.Yu. Olshanskii has obtained a negative answer on this question also.

F.18. THEOREM (A.Yu. Olshanskii [**OA1979**]). *There exists a torsion – free simple group G, whose proper subgroups are cyclic.*

Being simple, such a group cannot be polycyclic – by – finite, and it obviously satisfies the maximal condition for all subgroups.

It is not hard to prove that a group G satisfies the maximal condition for all subgroups if and only if every subgroup of G (and G itself) is finitely generated.

Just note that subgroups of finitely generated groups are not always finitely generated. One of the simplest examples confirming it is the following.

Let p be a prime and $A = \mathbb{Q}_p = \{ m/p^k \mid m, n \in \mathbb{Z} \}$ be an additive group of p – adic fractions. Clearly that the mapping ρ: $a \to pa$, $a \in A$, is an automorphism of A. Let G be a natural semidirect product of A and $< \rho >$. Then G is generated by the elements (1,1) and (0,ρ), but the subgroup A can not be finitely generated.

Thus, we see that the condition to be finitely generated is weaker than the condition **Max**. Since every finite group is finitely generated, this condition is a finiteness condition. The class of all finitely generated groups is one of the most important group – theoretical classes. With the condition to be a finitely generated group, it involves a lot of interesting and important results, which largely determined the development of group theory. One of the first major problems here were three problems posed by W. Burnside. In 1902, William Burnside [**BW1902**] wrote:

«A still undecided point in the theory of discontinuous groups is whether the order of a group may be not finite, while the order of every operation it contains is finite».

In modern terminology, the most general form of the problem is

«Can a finitely generated periodic group be infinite?»

Later, this problem was named the ***Generalized Burnside Problem.*** A negative solution of this problem was obtained in 1964.

F.19. THEOREM (E.S. Golod [**GE1964**]). *Let p be a prime and d be a positive integer. Then there exists a p – group G satisfying the following conditions:*

(i) G is infinite;

(ii) G is generated by d elements;

(iii) every subgroup of G which generated by d – 1 elements is finite;

(iv) G is residually finite;

(v) the orders of elements of G are not bounded.

Following the example of E.S. Golod, other examples of groups with similar properties were developed by other authors. Among these examples, it is worth noting the elegant and simple in construction example due to R.I. Grigorchuk [**GR1980**]. However, despite the simplicity of the construction, the Grigorchuk group is a bit sophisticated. Thus, many of the questions about the structure of the Grigorchuk group are not clarified yet.

However, W. Burnside was particularly interested in the question of whether or not a finitely generated group of finite exponent can be infinite.

We recall that a group G is ***bounded*** or ***has a finite exponent b*** if $g^{\mathbf{b}} = 1$ for each element g of a group G, and **b** is the smallest positive integer with this property.

We can state the Burnside Problem (as it has come to be known) in the following form. Let F_k be the free group with k free generators and let $\mathbb{B}(k,n) = F_k / F_k^n$. The group $\mathbb{B}(k,n)$ is called the ***k – generated Burnside group of exponent n***.

If G is a group of finite exponent n having k generators, then, clearly, G is isomorphic to some factor – group of the group $\mathbb{B}(k,n)$. This shows that the ***Burnside Problem*** can be reformulated in the following way:

is every group $\mathbb{B}(k,n)$ finite?

It is easy to see that groups of exponent 2 are elementary abelian, and so $\mathbb{B}(k,2)$ is a direct sum of k copies of the cyclic group of order 2. The question of the finiteness of the groups $\mathbb{B}(k,n)$ turned to be very difficult, even for small values of n. Thus, for example, it is still unknown if there can be an infinite group $\mathbb{B}(k,5)$. For other small values of n the situation is as follows.

F.20. THEOREM.

(i) $\mathbb{B}(k,3)$ *is finite for every positive integer k* (W. Burnside).

(ii) $|\mathbb{B}(k,3)| = 3^t$, *where* $t = k + \binom{k}{2} + \binom{k}{3}$ (F. Levi, B.L. van der Waerden [**LVW1933**]).

(iii) $\mathbb{B}(k,4)$ *is finite for every* $k \in \mathbb{N}$ (I.N. Sanov [**SI1940**]).

(iv) $|\mathbb{B}(k,4)| = 2^t$, *where* $\frac{1}{2}4^k \le t \le \frac{1}{2}(4+2\sqrt{2})^k$ (A. Mann [**MA1982**]).

(v) $|\mathbb{B}(k,6)| = 2^{\boldsymbol{a}}\, 3^{\boldsymbol{b}}$, $a = 1 + (k-1)\, 3^{\boldsymbol{t}}$, $t = k + \binom{k}{2} + \binom{k}{3}$, $b = 1 + (k-1)\, 2^{\boldsymbol{k}}$ (M. Hall [**HM1958**]).

P.S. Novikov and S.I. Adjan have obtained the solution of Burnside Problem for groups having finite exponent.

F.21. THEOREM (P.S. Novikov, S.I. Adjan [**NA1968[1]**] – [**NA1968[3]**]). *The group* $\mathbb{B}(k,n)$ *is infinite if* $k > 1$, *n is odd and* $n \ge 4381$.

Later in the book [**AS1979**], S.I. Adjan improved this result and proved that $\mathbb{B}(k,n)$ is infinite if $k > 1$, n is odd and $n \ge 665$. The entire book [**AS1979**] has been dedicated to the proof of this result. Using a different, namely geometrical, method, that allowed a significantly short proof, A.Yu. Olshanskii has proven that $\mathbb{B}(k,n)$ is infinite if $k > 1$, n is odd and $n > 10^{10}$ [**OA1982[1]**].

The case when $\mathbb{B}(k,n)$ has even exponent has been considered by S.V. Ivanov.

F.22. THEOREM (S.V. Ivanov [**IS1994**]). *The group $\mathbb{B}(k,n)$ is infinite if $k > 1$ and $n \geq 2^{48}$.*

Later I.G. Lysenok [**LI1996**], improving this result, has proven that the group $\mathbb{B}(k,n)$ is infinite if $k > 1$ and $n \geq 8000$.

Another variant of W. Burnside's original question is the ***Restricted Burnside Problem***, which may be stated as follows.

Are there only finitely many finite k – generator groups of exponent n?

W. Magnus called this problem the *Restricted Burnside Problem.*

This problem can be reformulated. Every finite group of exponent n having k generators is an epimorphic image of $\mathbb{B}(k,n)$. Consider the finite residual $\mathbb{B}(k,n)^{\mathfrak{F}}$ of the group $\mathbb{B}(k,n)$. Then every finite group of exponent n having k generators is an epimorphic image of the factor – group $\mathbb{B}(k,n)/\mathbb{B}(k,n)^{\mathfrak{F}} = \mathbb{R}(k,n)$. Then the Restricted Burnside Problem can be reformulated as follows:

is the group $\mathbb{R}(k,n)$ finite?

As mentioned above Burnside Problem and Generalized Burnside Problem were solved in the negative. It seemed natural that this problem is solved in the negative also. However, its surprisingly positive solution has been obtained by E.I. Zelmanov.

In 1959, A.I. Kostrikin in [**KAI1959**] has proven the finiteness of $\mathbb{R}(k,p)$ for every prime p. Later, however, it became clear that the proof has significant gaps. A.I. Kostrikin eliminated all the gaps, and published the complete proof in the book [**KAI1990**].

In 1956, Ph. Hall and G. Higman [**HH1956**] proved the following reduction theorem. If $n = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$ is a primary decomposition of positive integer n, then (subject to certain assumptions about finite simple groups) a positive solution of the Restricted Burnside Problem for exponent n follows from positive solutions for each prime – power factor $p_j^{k_j}$, $1 \le j \le m$. The classification of finite simple groups, which has been announced in the book [**GD1982**], implies that the assumptions made by Ph. Hall and G. Higman are valid. The last key steps have been completed by E.I. Zelmanov.

F.23. THEOREM.

(i) The group $\mathbb{R}(k,p^n)$ *is finite for every odd prime p and every positive integer n.* (E.I. Zelmanov [**ZI1990**]).

(ii) The group $\mathbb{R}(k,2^n)$ *is finite for every positive integer n.* (E.I. Zelmanov [**ZI1991**]).

(iii) The group $\mathbb{R}(k,t)$ *is finite for every positive integers k, t* (E.I. Zelmanov [**ZI1991**]).

We will not describe all the matters related to the finitely generated groups. Actually, it is not even possible, since the theory of finitely generated groups is very extensive, advanced and rich for many interesting and important results. It requires a separate presentation, and not in a single monograph. It concerns certain types of finitely generated groups. The problem of a full description of all finitely generated groups is very weighty, as the following result shows.

F.24. THEOREM (G. Higman, B.H. Neumann, H. Neumann [**HNN1949**]). *For each countable group G, there exists a group H, generated by two elements, which includes an isomorphic copy of G.*

We will meet finitely generated groups in our setting, so now we give only a few of the results that are connected to our main theme. The next two results are very useful.

F.25. THEOREM. *Let G be a finitely generated group. If H is a subgroup of G having finite index, then H is finitely generated.*

F.26. THEOREM (M. Hall [**HM1950**]). *Let G be a finitely generated group. Then the family of all subgroup of G having finite index n is finite for every positive integer n.*

This last result easily follows from **Theorem F.23**. Let G be a finitely generated group. More precisely, suppose that G has k generators, and n be a positive integer. Put

$\mathfrak{I}_n(G) = \{ H \mid H$ is a subgroup of G such that $|G:H| \leq n \}$.

If H is a subgroup of G of index n, then $|G:H^g| = n$ for every element $g \in G$. The subgroup $\mathbf{Core}_G(H) = \bigcap_{g \in G} H^g$ is normal in G and has finite index at most n!. It follows that $x^t \in \mathbf{Core}_G(H)$ for each element $x \in G$ and $t \leq n!$. In turn, it follows that the factor – group $G/\bigcap \mathfrak{I}_n(G) = L_n$ has finite exponent, say b_n. Clearly, G/L_n is residually finite. We have $L_n \cong \mathbf{B}(k,b_n)/V$ for some normal subgroup V of $\mathbf{B}(k,b_n)$. The fact that G/L_n is residually finite implies that $\mathbb{B}(k,b_n)^{\mathfrak{F}} \leq V$. Using **Theorem F.23**, we obtain that $\mathbb{B}(k,b_n)/\mathbb{B}(k,b_n)^{\mathfrak{F}}$ is finite, and hence $\mathbb{B}(k,b_n)/V \cong L_n$ is finite. Thus, we obtain **Theorem F.26**.

Let G be a finitely generated group. Then there is a free group F, having finite set of free generators, such that $G \cong F/K$ for some normal subgroup K. A group G is said to be ***finitely presented***, if there is a finite subset M of K such that $K = \langle M \rangle^F$. Every word $w \in M$ defined the equality $w = 1$, which valid for all elements w of G. In this case, we say that G ***is generated by a finite set of generators and it satisfies a finite set of defining relations.***

Finitely presented groups form a much narrower class of groups than any group with a finite number of generators. Thus, the latter form a set of cardinality of continuum, while, as simple set – theoretic considerations show, the set of finitely presented groups are only countable. Finitely presented groups arise in the consideration of various issues not only in algebra, but also in other mathematical disciplines. In particular, their role is quite significant in many geometric and topological problems. Again, we note that all of this is the subject of a separate consideration. Here we remark only the following important property of finitely presented groups.

F.27. THEOREM (R. Baer [BR1956]). *Let G be a group and K be a normal subgroup of G. Then:*

(i) if G is a finitely generated and the factor – group G/K is finitely presented, then K includes a finite subset M such that $K = \langle M \rangle^G$*;*

(ii) if K and G/K are finitely presented, then G is also finitely presented.

Chapter

3

On some associated with groups numerical invariants

The presence of certain numerical invariants associated with an algebraic structure often opens the doors to deep and detailed study of this structure. A striking example of it is vector spaces. They possess an essential numerical characteristic which is called the *dimension*. Groups also are associated with numerical characteristics, some of which in one way or another are analogs of the concept of the dimension of a vector space. These numerical invariants are not universal, and since they are introduced and employed in specific classes of groups, their nature is local. One of the earliest numerical characteristics is as follows.

Let G be a finitely generated group. Denote by $\mathbf{d}$(G) the smallest number of generators of the group G.

Here have a complete analogy with vector spaces: the dimension of a vector space is precisely the number of elements in a minimal system of generators. However, this analogy ends at this point. A group may have minimum sets of generators with a different number of elements. Besides, finitely generated groups can have non – finitely generated subgroups. Moreover, a finitely generated group G, whose subgroups are finitely generated (that is the group satisfying the maximal condition for all subgroups), can includes a subgroup H such that $\mathbf{d}$(H) > $\mathbf{d}$(G). Thus, the possessed by the dimension the property of homogeneity is not observed in groups.

Now we consider other numerical invariants, in varying degrees preserving the analogy with the notion of dimension. One of the first extensions of the concept of dimension was the concept of the R – rank of a module A over a (commutative) ring R. Since every abelian group is a module over the ring $\mathbb{Z}$ of integers, the concept of $\mathbb{Z}$ – rank (in the theory of abelian groups we use the term 0 – rank) has been resourcefully working in the theory of abelian groups.

Consider an analog of linearly independent subsets of a vector space.

Let G be an abelian group. A subset X of G consisting of elements of infinite order is said to be ***$\mathbb{Z}$* – independent** or simply ***independent*** if, given distinct elements $x_1, \ldots, x_n$ of X and integers $k_1, \ldots, k_n$, the relation $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n} = 1$ implies that $x_j = 1$ for all j, $1 \le j \le n$.

R.1. PROPOSITION. *Let G be an abelian group. If $X = \{ x_\lambda \mid \lambda \in \Lambda \}$ is a $\mathbb{Z}$ – independent subset of G, then $< X > = \mathbf{Dr}_{\lambda \in \Lambda} < x_\lambda >$ is a direct product of infinite cyclic subgroups. Conversely, if $H = \mathbf{Dr}_{\lambda \in \Lambda} < c_\lambda >$ where c_λ is an element of infinite order for every $\lambda \in \Lambda$, then the subset $\{ c_\lambda \mid \lambda \in \Lambda \}$ is $\mathbb{Z}$ – independent.*

Recall that a group G, which is decomposed into a direct product of infinite cyclic groups, is called ***free abelian*** group.

Zorn's Lemma implies that a $\mathbb{Z}$ – independent subset of an abelian group is always contained in some maximal $\mathbb{Z}$ – independent subset. By analogy with the vector space theory, we have the following fundamental result, which we record as the following proposition. The reader can find a proof of it in many excellent books (see, for example, this proof in [**FL1970**, Chapter III]).

R.2. PROPOSITION. *Let G be an abelian group. Then:*

(i) if G has an infinite $\mathbb{Z}$ *– independent subset, then every two maximal* $\mathbb{Z}$ *– independent subsets of G have the same cardinality;*

(ii) if G has finite maximal $\mathbb{Z}$ *– independent subset M, then each maximal* $\mathbb{Z}$ *– independent subset S of G is finite and* $|S| = |M|$*;*

(iii) if X is a maximal $\mathbb{Z}$ *– independent subset of G, then the factor – group* $G/ < X >$ *is periodic; conversely, if Y is a* $\mathbb{Z}$ *– independent subsets of G such that* $G/ < X >$ *is periodic, then Y is a maximal* $\mathbb{Z}$ *– independent subset of G.*

Let G be an abelian group. The cardinality of a maximal $\mathbb{Z}$ – independent subset of G is called the ***$\mathbb{Z}$ – rank*** or ***torsion – free rank*** of the group G, and is denoted by $\mathbf{r}_{\mathbf{Z}}(G)$. If G has a finite maximal $\mathbb{Z}$ – independent subset, then we will say that G ***has finite $\mathbb{Z}$ – rank***.

Since the concept of $\mathbb{Z}$ – rank is associated with elements of infinite order, the following statement looks natural.

R.3. PROPOSITION. *Let G be an abelian group. Then* $\mathbf{r}_{\mathbf{Z}}(G) = \mathbf{r}_{\mathbf{Z}}(G/\mathbf{Tor}\,(G))$.

A group G is said to be ***locally cyclic***, if every finite subset of G generates a cyclic subgroup.

The structure of locally cyclic groups is described in the following proposition.

R.4. PROPOSITION.

(i) The additive group of the field $\mathbb{Q}$ *of rational numbers has* $\mathbb{Z}$ *– rank 1.*

(ii) Let G be a torsion – free abelian group. Then $\mathbf{r}_{\mathbf{Z}}(G) = 1$ *if and only if G is a locally cyclic group.*

(iii) Let G be a torsion – free abelian group. Then $\mathbf{r}_{\mathbf{Z}}(G) = 1$ *if and only if G is isomorphic to some subgroup of the additive group of the field* $\mathbb{Q}$.

R.5. COROLLARY. *Let G be a torsion – free abelian group. Then* $\mathbf{r}_{\mathbf{Z}}(G) = k$ *if and only if G is isomorphic to some subgroup of* $A_1 \times \ldots \times A_k$ *where* A_j *is the additive group of the field* $\mathbb{Q}$, $1 \leq j \leq k$.

It follows from it that an abelian group G has finite $\mathbb{Z}$ – rank **r** if and only if G/**Tor**(G) is isomorphic to a subgroup of the additive group

$$\underbrace{\mathbb{Q}_{+} \oplus \ldots \oplus \mathbb{Q}_{+}}_{\mathbf{r}}$$

A.I. Maltsev in [**MAI1951**] called an abelian group G an ***abelian*** $\boldsymbol{A_1}$ ***– group*** if $\mathbf{r}_{\mathbf{Z}}(G)$ is finite.

This is a natural intention to consider some non – commutative analogs of $\mathbb{Z}$ – rank, which could be efficient in some classes of non – abelian groups.

A.I. Maltsev extended the notion of abelian A_1 – group to soluble groups.

Soluble group G is called a ***soluble*** $\boldsymbol{A_1}$ ***– group***, if G has a finite subnormal series whose factors are abelian A_1 – groups.

These groups also possess a finite subnormal series, whose factors are periodic or infinite cyclic. It was observed that the number of infinite cyclic factors is a group invariant. For the case of polycyclic – by – finite groups it was noted by K.A. Hirsch in his article [HKA1938].

For the case of polycyclic – by – finite groups the amount of such infinite cyclic factors was called the ***Hirsch number***.

The following statement gives us the basis for the introduction of a non – abelian analogue of $\mathbb{Z}$ – rank.

R.6. PROPOSITION. *Let G be a group and suppose that*

$$(\mathbf{a})\ \langle 1 \rangle = G_0 \lhd G_1 \lhd \ldots G_\alpha \lhd G_{\alpha+1} \ldots G_\gamma = G,$$

$$(\mathbf{b})\ \langle 1 \rangle = H_0 \lhd H_1 \lhd \ldots H_\beta \lhd H_{\beta+1} \ldots H_\delta = G$$

*are two ascending series of subgroups whose factors are either infinite cyclic or periodic. If the number of infinite cyclic factors of the series (**a**) is exactly **r**, then the number of infinite cyclic factors in series (**b**) is also **r**.*

This result shows that the number of infinite cyclic factors in the series in **Proposition R.6** is an invariant of a group.

A group G is said to have ***finite 0 – rank*** $\mathbf{r}_0(G) = \mathbf{r}$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly **r**. If G has an ascending series with periodic and infinite cyclic factors and the set of infinite cyclic factors are infinite, then we will say that group G ***has infinite 0 – rank.*** Otherwise we will say that G ***has no 0 – rank.***

Thus, it will be implicit in this chapter that the considered groups have reasonably well – behaved ascending series. We remark that this definition is a slight generalization of the traditional definition where one considered groups with a finite subnormal series whose factors are either infinite cyclic or periodic. In some papers, 0 – rank is called also the ***torsion – free rank*** of the group G.

For abelian groups $\mathbb{Z}$ – rank and 0 – rank are coincided. Therefore, further for abelian groups we will use the term 0 – rank instead of the term $\mathbb{Z}$ – rank.

Note some elementary properties of the 0 – rank.

R.7. PROPOSITION. *Let G be a group. Suppose that H is a subgroup of G and L is a normal subgroup of G. Then:*

(i) if G has finite 0 – rank, then H has finite 0 – rank and $\mathbf{r}_0(H) \leq \mathbf{r}_0(G)$;

(ii) if L is periodic, then $\mathbf{r}_0(G) = \mathbf{r}_0(G/L)$;

(iii) G has finite 0 – rank if and only if both L and G/L have finite 0 – rank; in this case

$$\mathbf{r}_0(G) = \mathbf{r}_0(L) + \mathbf{r}_0(G/L).$$

The first class (after abelian groups), in which we consider groups of finite 0 – rank, is the class of locally nilpotent groups. We have already noted that as for abelian groups, the set of all elements of finite orders of a locally nilpotent group G is a characteristic subgroup $\mathbf{Tor}(G)$ and the factor – group $G/\mathbf{Tor}(G)$ is torsion – free. By **Proposition R.7**, $\mathbf{r}_0(G) = \mathbf{r}_0(G/\mathbf{Tor}(G))$, so that, in this case, we need to consider only torsion – free locally nilpotent groups.

R.8. PROPOSITION (N.F. Sesekin [**SNF1953**]). *Let G be a torsion –free locally nilpotent group and suppose that G includes a normal abelian subgroup A such that* $\mathbf{r}_0(A) = \mathbf{k}$. *Then the* $\mathbf{k}^{th}$ *term of the upper central series of G includes A.*

We denote the nilpotency class of a group G by **ncl**(G). As usual, in the case of soluble groups, the structure of abelian subgroups plays a critical role. Next couple results illustrate it quite well.

R.9. PROPOSITION (N.F. Sesekin [**SNF1953**]). *Let G be a torsion –free hypercentral group and suppose that G includes a maximal normal abelian subgroup A such that* $\mathbf{r}_0(A) = \mathbf{k}$. *Then G is nilpotent and has finite 0 – rank at most* $\mathbf{k}(\mathbf{k} + 1)/2$. *Furthermore,* $\mathbf{ncl}(G) \leq 2\mathbf{k}$.

R.10. THEOREM (A.I. Maltsev [**MAI1951**]). *Let G be a torsion – free locally nilpotent group. Suppose that every abelian subgroup of G has a finite 0 – rank. Then G is a nilpotent group of finite 0 – rank. Furthermore, if A is a maximal normal abelian subgroup of G, and* $\mathbf{r}_0(A) = \mathbf{k}$, *then* $\mathbf{r}_0(G) \leq \mathbf{k}(\mathbf{k} + 1)/2$ *and* $\mathbf{ncl}(G) \leq 2\mathbf{k}$.

R.11. COROLLARY. *Let G be a torsion – free locally nilpotent group of finite 0 – rank. Then:*

(i) G has a finite subnormal series, whose factors are locally cyclic torsion – free groups;

(ii) G has a finite subnormal series, whose factors are infinite cyclic or periodic abelian groups with Chernikov Sylow p – subgroups for all prime p;

(iii) the Sylow p – subgroup of every periodic section of G are Chernikov for all prime p.

A group G is called ***polyrational*** if it has a finite subnormal series whose factors are torsion – free locally cyclic groups.

We observed in **Proposition R.4** that a torsion – free locally cyclic group is isomorphic to a subgroup of $\mathbb{Q}_+$. Therefore a torsion – free locally cyclic group is also called a ***rational*** group. This makes the above definition entirely justified.

Investigating the hypercentral $\mathfrak{F}$ – perfect groups, S.N. Chernikov was noted that a hypercentral torsion – free group has a central ascending series with locally cyclic torsion – free factors, and the length of this series is an invariant of such group [**CSN1950[1]**]. A little bit later, he observed that in a group with finite subnormal series with locally cyclic torsion – free factors (rational series), the length of this series is an invariant of such group. This number he called the ***rational rank***. This number is exactly 0 – rank of polyrational group.

Now we consider a slightly weaker definition of the groups of finite 0 – rank. More precisely, we come to the more usual definition.

R.12. THEOREM. *Let G be a group of finite 0 – rank. Then G has a finite series of normal subgroups, every factor of which is either a periodic group or torsion – free nilpotent polyrational group.*

R.13. COROLLARY. *Let G be a group of finite 0 – rank. Then G has a finite subnormal series, every factor of which is either a periodic group or infinite cyclic group.*

For locally generalized radical groups we have the following result.

R.14. PROPOSITION (M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov [**DKP2007**]). *Let G be a group. Then the following statements are equivalent:*

*(i) G has an ascending series whose factors are either infinite cyclic or locally finite and the number of infinite cyclic is exactly **r**;*

*(ii) G is a generalized radical group of finite 0 – rank **r**;*

*(iii) G is a locally generalized radical group of finite 0 – rank **r**.*

Now we can describe the structure of locally generalized radical groups of finite 0 – rank.

R.15. THEOREM (M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov [**DKP2007**]). *Let G be a locally generalized radical group of finite 0 – rank. Then G has normal subgroups $T \leq L \leq K \leq S \leq G$ such that*

(i) T is locally finite and G/T is soluble – by – finite;

(ii) L/T is torsion – free nilpotent group;

(iii) K/L is finitely generated torsion – free abelian group;

(iv) G/K is finite and S/K is soluble radical of G/K.

Moreover, if $\mathbf{r_0}(G) = \mathbf{r}$, then there are the functions $\mathbf{f}_1$, $\mathbf{f}_2$ such that $|G/K| \leq \mathbf{f}_2(\mathbf{r})$ and $\mathbf{dl}\,(S/T) \leq \mathbf{f}_1(\mathbf{r})$.

R.16. COROLLARY. *Let G be a locally generalized radical group of finite 0 – rank. Then G/***Tor** *(G) includes normal polyrational subgroups of finite index.*

R.17. COROLLARY. *Let G be a polyrational group. Then G has the normal subgroups* $L \le K \le G$ *such that L is torsion – free nilpotent, K/L is finitely generated torsion – free abelian and G/K is finite. Furthermore, if* $\mathbf{r_0}(G) = \mathbf{r}$*, then* $|G/K| \le \mathbf{f}_3(\mathbf{r})$ *and* $\mathbf{dl}\,(G) \le \mathbf{f}_2(\mathbf{r})$ *for some functions* $\mathbf{f}_1$ *and* $\mathbf{f}_3$.

Now consider local properties of 0 – rank.

R.18. PROPOSITION (M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov [**DKP2007**]). *Let G be a locally generalized radical group. Suppose that there exists a positive integer r such that every finitely generated subgroup of G has finite 0 – rank at most* **r**. *Then G has 0 – rank at most* **r**. *In particular, G is a generalized radical group.*

We present a useful summary of this local theorem. Along the way, note the following important results.

R.19. PROPOSITION. *Let* $G = \mathbf{Cr}_{\lambda \in \Lambda} G_\lambda$ *where* G_λ *is a finite group for each* $\lambda \in \Lambda$. *If there is a positive integer t such that* $|G_\lambda| \le t$ *for every* $\lambda \in \Lambda$, *then G is a locally finite group.*

R.20. PROPOSITION (O.H. Kegel, B.A.F. Wehrfritz [**KW1973**, Proposition 1.K.2]). *Let $\mathfrak{X}$ be a class of groups such that for groups $X \geq S \geq Y$ with $X, Y \in \mathfrak{X}$ the finiteness of either of the indices $|X:S|$ and $|S:Y|$ implies that S belongs to $\mathfrak{X}$. If the group G has a local system $\mathfrak{L}$ consisting of finitely generated subgroups such that each subgroup $L \in \mathfrak{L}$ has a subgroup of index at most **n** belonging to $\mathfrak{X}$, then G has a subgroup of index at most **n** which has a local system consisting of finitely generated $\mathfrak{X}$ – subgroups.*

Now we are on the position to formulate the following generalization of **Proposition R.18**.

R.21. THEOREM. *Let G be a group and suppose that G satisfies the following conditions:*

*(i) for every finitely generated subgroup L of G the factor – group L/**Tor** (L) is a locally generalized radical group;*

*(ii) there is a positive integer **r** such that $\mathbf{r}_0(L) \leq \mathbf{r}$ for every finitely generated subgroup L.*

*Then G/**Tor** (G) includes a normal soluble subgroup D/**Tor** (G) of finite index. Moreover, G has finite 0 – rank **r** and there is a functions $\mathbf{f}_4$ such that $|G/D| \leq \mathbf{f}_4(\mathbf{r})$.*

The above results show that 0 – rank is a substantial numerical invariant of a group. It is so significant that we were able to get a quite clear description of the very wide class of groups with a finite 0 – rank. However, this numerical invariant can only work in non – periodic groups ($\mathbf{r}_0(G) = 0$ whenever a group G is periodic). Farther we will consider another important numerical invariant, which can operate in non – periodic as while as in periodic groups. It is associated with the concept of 0 – rank and in some ways looks a little more general. Its roots are also in vector space dimension, but this connection does not seem so obvious. As before, we start with abelian groups. We consider in them the concept of p – rank and show how it is transformed into the concept of section p – rank.

For periodic abelian groups, the following numerical invariant, which is also based on the concept of dimension of vector space, was introduced.

Let p be a prime, and let n be a positive integer. If P is an abelian p – group then the ***n – layer*** of P is the subgroup

$$\Omega_n(P) = \{ a \in P \mid |a| \text{ divides } p^n \}.$$

It is clear that, $\Omega_{n+1}(P)/\Omega_n(P)$ is an elementary abelian p –group for each n, which can therefore be thought of as a vector space over the prime field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. In particular, $\Omega_1(P)$ is such a vector space.

Let p be a prime and P an abelian p – group. The ***p – rank*** of P is defined to be the dimension of the vector space $\Omega_1(P)$ over $\mathbf{F}_p$. More generally, if G is an arbitrary abelian group then we define the ***p – rank*** of G to be the p – rank of $\mathbf{Tor}_p(G)$. The p – rank of G is denoted by $\mathbf{r}_p(G)$.

In particular, if $\mathbf{Tor}_p(G) = \langle 1 \rangle$, then $\mathbf{r}_p(G) = 0$. It is clear that p – rank of $\mathbf{C}_p \times \mathbf{Z}$ is 1 and that p – rank of the factor – group $(\mathbf{C}_p \times \mathbf{Z})/p\mathbf{Z}$ is 2. Thus p – rank can increases when we pass to factor – groups. From the definition given above it follows that if G is a finite elementary abelian p – group is isomorphic to a direct product of n copies of $\mathbf{C}_p$, the cyclic group of order p, then $\mathbf{r}_p(G) = n$.

The following characterization of p – rank served as the base for the concept of the section p – rank.

R.22. PROPOSITION. *Let P be an abelian p – group for some prime p. Then* $\mathbf{r}_p(P) = \mathbf{r}$ *is finite if and only if every elementary abelian section U/V of P is finite,* $\mathbf{r}_p(U/V) \leq \mathbf{r}_p(P)$*, and there is an elementary abelian section A/B of P such that* $\mathbf{r}_p(A/B) = \mathbf{r}$*.*

Note the following standard futures of the abelian groups having finite p – rank.

R.23. PROPOSITION. *Let P be an abelian p – group for some prime p. Then* $\mathbf{r}_p(P)$ *is finite if and only if P is a Chernikov group.*

R.24. COROLLARY. *Let P be an abelian p – group for some prime p. Then P is a Chernikov group if and only if* $\mathbf{\Omega}_1(P)$ *is finite.*

We can observe that there is a clear split in the theory of abelian groups: 0 – rank works in torsion – free groups while p –rank is used in periodic groups. Nevertheless, D.J.S. Robinson [RD1968, 6.1], just mechanically combining the concepts of 0 – rank and p – rank, has introduced the class $\mathfrak{A}_0$.

An abelian group A ***belongs to the class*** $\mathfrak{A}_0$ if and only if $\mathbf{r}_0(A)$ is finite and $\mathbf{r}_p(A)$ are finite for all primes p.

He extends it to soluble groups in the traditional way.

A soluble group G ***belongs to the class*** $\mathfrak{S}_0$ if and only if G has a finite subnormal series, every factor of which is abelian $\mathfrak{A}_0$ – group.

Now we introduce a universal concept, which works for both periodic and non – periodic groups and incorporated the important properties of 0 – rank and p – rank. As noted above, it arises from the properties of p – rank considered in **Proposition R.22**.

Let p be a prime. We say that a group G has ***finite section p – rank*** $\mathbf{sr}_p(G) = r$ if every elementary abelian p – section of G is finite of order at most p^r and there is an elementary abelian p – section A/B of G such that $|A/B| = p^r$.

As it easily seen, it is always the case for an abelian group G that $\mathbf{r}_p(G) \leq \mathbf{sr}_p(G)$. However, p – rank and section p – rank in general do not coincide for abelian groups. It is clear that p – rank of $\mathbb{C}_p \times \mathbb{Z}$ is 1. But this group has an elementary abelian p – section $(\mathbb{C}_p \times \mathbb{Z})/p\mathbb{Z}$ of order p^2. It is easy to see that section p – rank of $\mathbb{C}_p \times \mathbb{Z}$ is precisely 2.

Observe that if a group G has an element g of infinite order, then G has a section $\langle g \rangle / \langle g^p \rangle$ of order p for every prime p. Hence if G is not periodic, then $\mathbf{sr}_p(G) \geq 1$ for every prime p.

If a group G has an element g of order p for prime p, then G has the section $\langle g \rangle / \langle 1 \rangle$ of order p. Hence, if $p \in \Pi(G)$, then $\mathbf{sr}_p(G) \geq 1$. The equality $\mathbf{sr}_p(G) = 0$ means that G is a periodic group containing no p – elements.

The following elementary result enables us to easily determine a number of examples of groups of finite section p – rank.

R.25. PROPOSITION. *Let p be a prime and let G be a group.*

(i) Suppose that G has finite section p – rank. If K is a subgroup of G and H is a normal subgroup of K, then $\mathbf{sr}_p(K/H) \le \mathbf{sr}_p(G)$.

(ii) Suppose that G has finite section p – rank. If H is a normal subgroup of G, then $\mathbf{sr}_p(G) \le \mathbf{sr}_p(H) + \mathbf{sr}_p(G/H)$.

(iii) Suppose that G has finite section p – rank. If H is a normal periodic subgroup of G such that $p \notin \Pi(H)$, *then* $\mathbf{sr}_p(G/H) = \mathbf{sr}_p(G)$.

(iv) Let H be a normal subgroup of G. If H and G/H have finite section p – ranks, then G has finite section p – rank.

As the cyclic group of order 4 readily shows, we does not need equality in part (ii) of this proposition. It follows from this proposition that the property of having finite section p – rank is closed under taking subgroups and homomorphic images. Part (iv) of this proposition shows that a Chernikov group is necessarily a group of finite section p – rank for each prime p.

The following result allows us in general to characterize the abelian groups with finite section p – rank. We first note that a Sylow p – subgroup of a group G is defined to be a maximal p –subgroup of G. Note that for an abelian group G, the Sylow p – subgroup is unique and it is precisely the p –component $\mathbf{Tor}_p(G)$.

R.26. PROPOSITION. *Let A be an abelian group and p be a prime. Then A has finite section p – rank if and only if the Sylow p – subgroup P of A is Chernikov and the 0 – rank of A is finite. Moreover,* $\mathbf{sr}_p(A) = \mathbf{sr}_p(P) + \mathbf{r}_0(A)$.

The next natural step is the study of locally nilpotent groups having finite section p – rank. The final result here is almost the same as it was in abelian case, but it requires much serious efforts to get. The first step here is investigation of the periodic locally nilpotent groups having finite section p – rank. It is clear, that this consideration is reduced to the consideration of locally nilpotent p – groups (or, what is equivalent, locally finite p –groups). The case of nilpotent groups has some specific sides which valid not to only for groups of finite section p – rank. In passing, we consider some useful technical results, which are valid not only for nilpotent groups.

R.27. PROPOSITION (O. Grün [**GO1935**]). *Let G be a group. Suppose that $\zeta_1(G) \neq \langle 1 \rangle$ and $\zeta_2(G) \neq \zeta_1(G)$. If $g \in \zeta_2(G) \setminus \zeta_1(G)$, then the mapping ξ_g, defined by the rule $\xi_g(x) = [g,x]$, $x \in G$, is an endomorphism of G. Moreover, **Im** $(\xi_g) = [g,G]$, **Ker** $(\xi_g) = \mathbf{C}_G(g)$, so that $[g,G]$ = **Im** $(\xi_g) \cong G/$**Ker** $(\xi_g) = G/\mathbf{C}_G(g)$. Furthermore, if $g^k \in \zeta(G)$, then $[g,G]^k = \langle 1 \rangle$.*

R.28. PROPOSITION (L.A. Kurdachenko [**KLA1984**]). *Let G be a hypercentral group and let A be a normal abelian p – subgroup of G for some prime p. If $G/\mathbf{C}_G(A)$ does not include the subgroups of index p, then $A \leq \zeta(G)$.*

R.29. COROLLARY (S.N. Chernikov [**CSN1946**]). *Let G be a hypercentral group. If G is $\mathfrak{F}$ – perfect, then its center includes* **Tor** *(G). In particular, $\mathfrak{F}$ – perfect periodic hypercentral group is abelian.*

R.30. PROPOSITION. *Let G be a nilpotent Chernikov group. Then:*

(i) G includes a finite normal subgroup F such that G/F is divisible;

(ii) G is central – by – finite.

Moreover, we have

R.31. PROPOSITION. *Let G be a periodic nilpotent group. Then the center of G includes the greatest* $\mathfrak{F}$ *– perfect subgroup of G.*

Now we are on the position to proceed with description of nilpotent groups of finite section p – rank.

R.32. PROPOSITION. *Let p be a prime and let P be a nilpotent p – group. Then* $\mathbf{sr}_p(P)$ *is finite if and only if P is a central – by – finite Chernikov group.*

Naturally, next step is to generalize **Proposition R.32** to locally finite p – groups. Here we will need some information about periodic groups of authomorphisms of Chernikov groups.

R.33. THEOREM (R. Baer [**BR1955**]). *Let G be a Chernikov group and A be a periodic group of automorphisms of G. Then A is a Chernikov group. If G is abelian, then A is finite.*

R.34. THEOREM (S.N. Chernikov [**CSN1950**]). *Let G be a locally finite p – group for some prime p. If every abelian subgroup of G is Chernikov, then G is Chernikov.*

R.35. COROLLARY. *Let p be a prime and let G be a locally finite p – group. Then G has a finite section p – rank if and only if G is a hypercentral Chernikov group.*

Observe that unlike in the nilpotent case, the center of locally finite Chernikov p – group needs not to include the divisible part. The simplest example of this is the locally dihedral group. Let K be a Prüfer 2 – group. Then K has an automorphism $\iota: x \to x^{-1}$, for $x \in K$, and the natural semidirect product $G = K \rtimes \langle \iota \rangle$ is a Chernikov 2 – group with a finite center.

There are similar examples for each prime p. Let

$$K_j = \langle a_{j,n} \mid a_{j,1}^p = 1, a_{j,n+1}^p = a_{j,n}, n \in \mathbf{N} \rangle$$

be a Prüfer p – group for $1 \leq j \leq p - 1$, and let $A = K_1 \times \ldots \times K_{p-1}$.

It is possible to show that A has an automorphism χ defined by the rule:

$$\chi(a_{j,n}) = \begin{cases} a_{j,n}a_{j-1,n}, & \text{if } j \geq 2, n \geq 1, \\ a_{j,n}, & \text{if } j = 1, n = 1, \\ a_{j,n}a_{p-1,n-1}, & \text{if } j = 1, n \geq 1. \end{cases}$$

The natural semidirect product $G = K \rtimes \langle \chi \rangle$ is easily seen to be a Chernikov p – group every upper central factor of whose has order p.

Next step is a general description of locally nilpotent groups having finite section p – rank. As for the abelian case, we have here very close connection with the 0 – rank.

R.36. THEOREM. *Let p be a prime and let G be a locally nilpotent group. Then G has finite section p – rank if and only if the Sylow p –subgroup P of G is Chernikov and 0 – rank of G is finite. Moreover,* $\mathbf{sr}_p(G) = \mathbf{sr}_p(P) + \mathbf{r}_0(G)$.

We note that the equality $\mathbf{sr}_p(G) = \mathbf{r}_0(G)$ does not valid not only for torsion – free soluble groups, but even for polyrational groups. The following example justifies this fact.

Let p be a prime number and let

$$A = \mathbf{Q}_p = \{ m/p^n \mid m, n \in \mathbf{Z} \}$$

be the additive group of p – adic rational numbers. Since A = pA, the mapping ρ: A → A, defined by the rule $\rho(a) = (1/p)\, a$, $a \in A$, is an automorphism of A. Clearly, this automorphism has infinite order. Consider a natural semidirect product $G = A \rtimes \langle \rho \rangle$, that is the set of pair (a,ρ^k), $a \in A$, $k \in \mathbf{Z}$, the multiplication on which is defined by the following rule:

$$(a,\rho^k)(b,\rho^t) = (a + p^k b, \rho^{k+t}),\ a, b \in A,\ k, t \in \mathbf{Z}.$$

A subgroup A is normal in G and the factor – group G/A is infinite cyclic, so that G is a polyrational group, and $\mathbf{r}_0(G) = 2$. A group G is soluble and finitely generated (it is generated by the pairs (1,1) and $(0,\rho)$). However, G has no elementary abelian section of order p^2.

Now consider locally finite groups of finite section p – rank for some prime p. If G is such a group, then it follows from **Corollary R.35** that every p – subgroup (and hence, every Sylow p – subgroup) of G is Chernikov. Here, by a Sylow p – subgroup we shall simply mean a maximal p – subgroup. However, as can be seen in [**DMR1994**], in infinite groups there are many different variations of the term «*Sylow theory*». Indeed, the locally finite groups of finite section p – rank are precisely the locally finite groups whose Sylow p – subgroups are Chernikov. We remark at once that the Sylow p – subgroups of such groups, in general, are not isomorphic. Indeed, locally finite groups with Chernikov Sylow p – subgroups can possess some badly behaved Sylow p –subgroups. However, in such groups, it is possible to find certain «nice» Sylow p – subgroups. Following the book [**DMR1994**], we will call them Wehrfritz p – subgroups, who first showed their existence in locally finite groups whose Sylow p – subgroups are Chernikov.

Let p be a prime and G be a locally finite group. A maximal p – subgroup P of G is called *a* ***Wehrfritz p – subgroup*** if P includes an isomorphic copy of every p – subgroup of G.

The first question here is about existence of Wehrfritz p –subgroups. For locally finite groups, whose Sylow p – subgroups are Chernikov, the answer is positive.

R.37. THEOREM (B.A.F. Wehrfritz [**WB1969**]). Let *G be a locally finite group whose p – subgroups are Chernikov for some prime p. Then G has Wehrfritz p – subgroups and every finite p – subgroup lies in at least one of these.*

This theorem allows obtaining the following result.

R.38. THEOREM. *Let G be a locally finite group and let p be a prime. The following are equivalent:*

(i) G has finite section p – rank;

(ii) the p – subgroups of G are Chernikov;

(iii) the abelian subgroups of G have finite section p – rank.

Imposing some natural restrictions, we come to the following result.

R.39. THEOREM (M.I. Kargapolov [**KMI1961**]). *Let G be a locally finite group of finite section p – rank for some prime p. Then the factor – group* $G/\mathbf{O}_{p',p}(G)$ *is finite if and only if every simple section of G, containing elements of order p, is finite.*

R.40. COROLLARY. *Let G be a periodic locally soluble group of finite section p – rank for the prime p. Then* $G/\mathbf{O}_{p'}(G)$ *is a Chernikov group.*

R.41. COROLLARY (S.N. Chernikov [**CSN1960**]). *Let G be a periodic locally soluble group. If the Sylow p – subgroup of G are finite for some prime p, then* $G/\mathbf{O}_{p'}(G)$ *is finite. In particular, if the Sylow p – subgroup of G are finite for all prime p, then G is residually finite.*

The above results allow us to obtain description of the structure of locally generalized radical groups of finite section p – rank.

R.42. THEOREM (M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov [**DKP2007**]). *Let G be a locally generalized radical group of finite section p – rank* $\mathbf{r}_p$ *for some prime p. Then G has finite 0 – rank at most* $2\mathbf{r}_p$*. Furthermore, G has normal subgroups*

$$T \leq L \leq K \leq S \leq G$$

such that

(i) T is a locally finite subgroup, whose Sylow p – subgroups are Chernikov, and G/T is soluble – by – finite;

(ii) L/T is a torsion – free nilpotent group;

(iii) K/L is a finitely generated torsion – free abelian group;

(iv) G/K is finite and S/K is the soluble radical of G/K.

Moreover, there are functions $\boldsymbol{f}_4$, $\boldsymbol{f}_5$ *such that* $|G/K| \leq \boldsymbol{f}_4(\boldsymbol{r}_p)$, $\boldsymbol{dl}\,(S/T) \leq \boldsymbol{f}_5(\boldsymbol{r}_p)$.

R.43. COROLLARY. *Let G be a locally generalized radical group of finite section p – rank* $\boldsymbol{r}_p$ *for some prime p. Suppose that every simple section of G containing elements of order p is finite. Then G has normal subgroups* $Q \leq T \leq L \leq K \leq S \leq G$ *such that Q is a locally finite p' –subgroup, T/Q is a Chernikov group whose divisible part is a p –group, L/T is torsion – free nilpotent, K/L is finitely generated torsion – free abelian, G/K is finite and S/K is soluble. Furthermore, G has finite 0 – rank* $\boldsymbol{r}_0(G) \leq 2\boldsymbol{r}_p$. *Moreover,* $|G/K| \leq \boldsymbol{f}_4(\boldsymbol{r}_p)$, $\boldsymbol{dl}\,(S/T) \leq \boldsymbol{f}_5(\boldsymbol{r}_p)$ *for some functions* $\boldsymbol{f}_4$ *and* $\boldsymbol{f}_5$.

R.44. COROLLARY. *Let G be a locally radical group of finite section p – rank* $\boldsymbol{r}_p$ *for some prime p. Then G has normal subgroups* $Q \leq T \leq L \leq K \leq G$ *such that Q is a periodic locally soluble p' – subgroup, T/Q is a soluble Chernikov group whose divisible part is a p – group, L/T is torsion – free nilpotent, K/L is finitely generated torsion – free abelian, and G/K is finite soluble group such that* $|G/K| \leq \boldsymbol{f}_4(\boldsymbol{r}_p)$ *and* $\boldsymbol{dl}\,(G/T) \leq \boldsymbol{f}_5(\boldsymbol{r}_p)$ *for some functions* $\boldsymbol{f}_4$ *and* $\boldsymbol{f}_5$. *Furthermore, G has finite 0 – rank* $\boldsymbol{r}_0(G) \leq 2\boldsymbol{r}_p$.

Now it is naturally to consider the case when a group has finite section p – rank for all prime p.

Let G be a group. We say that a group G has a ***finite section rank*** if $\mathbf{sr}_p(G)$ is finite for each prime number p.

We can slightly concretize this definition. Let σ be a function from the set $\mathbf{P}$ of all primes in $\mathbf{N}_0$. We say that a group G has a ***finite section rank σ***, if $\mathbf{sr}_p(G) = \sigma(p)$ for every prime p.

D.J.S. Robinson in the paper [**RD1972**, 9.3] introduced the following class of groups.

A group G is said to have ***finite abelian sectional rank*** if it has no infinite elementary abelian p – sections for any prime p.

It is clear that if a group G has finite section rank σ for some function $\sigma: \mathbf{P} \to \mathbf{N}_0$, then G has finite abelian sectional rank. Hence, the groups having finite section rank are some restricted kinds of the groups of finite abelian sectional rank. In reality, we observe that these cases often coincide. In particular, it takes place for abelian groups.

R.45. PROPOSITION. *Let G be an abelian group. Suppose that every elementary abelian section of G is finite. Then there exists a function* $\sigma : \mathbf{P} \to \mathbf{N}_0$ *such that G has finite section rank* σ*. Moreover* $\sigma(p) = \mathbf{r}_p(G) + \mathbf{r}_0(G)$ *for every prime p. In particular, if G is periodic,* $\sigma(p) = \mathbf{r}_p(G)$ *for every prime p. If G is torsion – free, then* $\sigma(p) = \mathbf{r}_0(G)$ *for each prime p.*

Note, that the study of the groups of finite section rank cannot be reduced to the simple corollaries of the above results. There are some interesting differences. As in previous case, the study of such groups naturally splits into two parts: the consideration of the maximal normal periodic subgroup **Tor**(G), and the consideration of the factor – group G/**Tor**(G). In the case when G is a locally generalized radical group, the subgroup **Tor**(G) is locally finite. Consequently, we must begin our study with the locally finite case. It is the main case here, it gives us an opportunity to advance.

R.46. THEOREM. *Let G be a locally finite group. The following assertions are equivalent:*

(i) G has finite section rank $\boldsymbol{\sigma}$*;*

(ii) each p – subgroup of G is Chernikov for every prime p;

(iii) if A is an arbitrary abelian subgroup of G, then p – rank of A is finite for all prime p;

(iv) if A is an arbitrary abelian subgroup of G, then section p –rank of A is finite for all prime p.

Moreover, if G has finite section rank $\boldsymbol{\sigma}$*, then* $\mathbf{r}_p(A) \leq \boldsymbol{\sigma}(p)$ *for every abelian subgroup A.*

The following result is an important property of locally finite groups of finite section rank.

R.47. THEOREM (V.V. Belyaev [**BVV1981**]). *Let G be a locally finite group. If G has finite section rank, then G is almost locally soluble.*

Using this result, we can show the following structural result, due to M. I. Kargapolov [**KMI1961**], in the following generalized form.

R.48. THEOREM (M.I. Kargapolov [**KMI1961**]). *Let G be a locally finite group of finite section rank. Then G includes a normal divisible abelian subgroup R such that G/R is residually finite and the Sylow p – subgroup of G/R are finite for each prime p.*

Note the following simple but useful property of groups of finite section rank.

R.49. PROPOSITION. *Let G is a locally finite group of finite section rank. If G is bounded, then G is finite.*

We will need the following result on finite groups.

We recall that a finite group G is called ***semisimple***, if G does not include non – trivial normal abelian subgroups.

A normal subgroup H of a finite group G is called ***completely reducible*** if H is a direct product of simple groups.

Every finite semisimple group G has a non – trivial maximal normal completely reducible subgroup. This subgroup is called the ***completely reducible radical*** of G (see, for example, [**KAG1967**, § 61]).

R.50. PROPOSITION. *Let G be a finite group of section 2 – rank* ***d****. Then G has normal subgroups $R \le S \le V \le G$ such that R is a soluble radical of G, S/R is a direct product at most d finite simple non –abelian groups, V/S is soluble and* $|G/V| \le$ ***d!****.*

R.51. COROLLARY. *Let G be a finite group of section 2 – rank* ***d****. Then the number of non – abelian composition factors of G is at most* ***d*** + ***d!****.*

As a corollary, we can mention the following property of groups of finite section rank.

R.52. COROLLARY. *Let G be a group of finite section rank. Then G includes a characteristic subgroup L of finite index whose finite factor – groups are soluble.*

Using the above results, we come to the following description of the structure of locally generalized radical groups of finite section rank.

R.53. THEOREM. *Let G be a locally generalized radical group of finite section rank. Then G has finite 0 – rank, moreover* $\mathbf{r}_0(G) \leq 2\mathbf{t}$ *where* $\mathbf{t} = \mathbf{min}\{\, \mathbf{sr}_p(G) \mid p \in \mathbb{P}\}$. *Furthermore, G has normal subgroups*

$$D \leq T \leq L \leq K \leq S \leq G$$

satisfying the following conditions:

(i) T is periodic and almost locally soluble;

(ii) Sylow p – subgroups of G are Chernikov for all primes p;

(iii) D is a divisible abelian subgroup;

(iv) Sylow p – subgroups of T/D are finite for all primes p, and T/D is residually finite;

(v) L/T is nilpotent and torsion – free;

(vi) K/L is abelian torsion – free and finitely generated;

(vii) G/K is finite and $|G/K| \leq \mathbf{f}_4(\mathbf{t})$ *for some function* $\mathbf{f}_4$;

(viii) S/K is soluble and $\mathbf{dl}\,(S/T) \leq \mathbf{f}_5(\mathbf{t})$ *for some function* $\mathbf{f}_5$.

In particular, G is generalized radical group.

R.54. COROLLARY. *Let G be a locally generalized radical group of finite section rank. Then G is (locally soluble) – by – soluble – by – finite.*

R.55. COROLLARY. *Let G be a locally (soluble – by – finite) group of finite section rank. Then G is almost locally soluble.*

The next theorem establishes connections between groups whose all elementary abelian sections are finite and groups of finite section rank.

R.56. THEOREM. *Let G be a generalized radical group. Suppose that every elementary abelian section of G is finite. Then there exists a function $\sigma: \mathbf{P} \to \mathbf{N}_0$ such that G has finite section rank σ.*

If A is a vector space of finite dimension **k** over a field F, and B is a subspace of A, then it is well – known that B is finite dimensional and that the dimension of B is at most **k**. Similarly, there is a well – known consequence of the structural theorem for finitely generated abelian groups. Namely, if G is an abelian group with **k** generators, and B is a subgroup of G, then B is finitely generated and has at most **k** generators. Thus a subgroup H of a finitely generated abelian group G is also finitely generated, and the minimal number of generators of H is at most the minimal number of generators of G. However, as it well – known, for non – abelian groups this sort of subgroup inheritance could not exists. For example, the standard restricted wreath product $\mathbf{Z}\ \mathbf{wr}\ \mathbf{Z}$ of two infinite cyclic groups is a 2 – generator group with its base group is infinitely generated. Thus, a subgroup of a finitely generated group need not even be finitely generated.

As we have seen, and as it is well – known, a subgroup of a finitely generated nilpotent group is finitely generated. However, even in this case, it is possible to have a subgroup with more generators than the original group has, as the group $\mathbf{C_p}$ **wr** $\mathbf{C_p}$, for p an odd prime shows. This group is 2 – generated but its base group is p – generated.

The group G has ***finite special rank*** $\boldsymbol{r}$*(G)* = $\boldsymbol{r}$ if every finitely generated subgroup of G can be generated by **r** elements and **r** is the least positive integer with this property.

The general concept of special rank (and also the term special rank) has been introduced by A.I. Maltsev [**MAI1948**]. In the paper [**BR1966**], R. Baer instead of the term «special rank» used the term «***Prüfer rank***». In the paper [**PH1924**], H. Prüfer defined a group of rank 1. However, it is not possible to learn from this article any roots of the concept of special rank. It should also be noted that this term was not employed in the theory of abelian groups. The basic concepts of p – rank and 0 – rank are employed there. We shall denote the class of groups of finite special rank by $\mathfrak{R}$, and for each positive integer n we shall denote by $\mathfrak{R}_n$ the class of groups of special rank at most n. It is clear that

$\mathfrak{R} = \bigcup_{n \in \mathbb{N}} \mathfrak{R}_n$.

We shall make some elementary observation concerning groups of finite special rank.

R.57. PROPOSITION. *Let G be a group, H be a subgroup of G and L be a normal subgroup of G. Then:*

(i) if G has finite special rank, then H has finite special rank, and $\mathbf{r}(H) \leq \mathbf{r}(G)$*;*

(ii) if G has finite special rank, then G/L has finite special rank, and $\mathbf{r}(G/L) \leq \mathbf{r}(G)$*;*

(iii) if L and G/L have finite special rank, then G has a finite special rank, and, moreover, $\mathbf{r}(G) \leq \mathbf{r}(L) + \mathbf{r}(G/L)$*.*

Observe that special rank has no additive property. Thus, in general, it is not true that $\mathbf{r}(G) = \mathbf{r}(L) + \mathbf{r}(G/L)$. There is a variety of easy supporting examples. For instance, if p is a prime and $G = \langle g \rangle$ is a cyclic group of order p^2, then G is of special rank 1. Furthermore, $L = \langle g^p \rangle$ and $G/L \cong \mathbf{C}_p$ also are of special rank 1.

In particular, **Proposition R.57** shows that the property of having finite special rank at most **r** is closed under taking subgroups and factor – groups. Furthermore, the property of having finite special rank is closed under taking extensions. We also note that if every finitely generated subgroup of G has finite special rank at most **r** then G has special rank at most **r**. Also we remark that the class of groups of finite special rank is countable recognizable.

The class of groups of finite special rank is not locally closed. However, the direct product of countable many copies of a cyclic group of prime order p is locally of finite special rank, but is not itself of finite special rank. Nevertheless, the following property is very useful.

R.58. PROPOSITION. *Let G be a periodic group and suppose that* $G = \mathbf{Dr}_{n \in \mathbb{N}} G_n$*. If there is a positive integer* $\mathbf{r}$ *such that* $\mathbf{r}(G_n) \leq \mathbf{r}$ *for each* $n \in \mathbb{N}$ *and* $\varnothing = \Pi(G_n) \cap \Pi(G_k)$ *whenever* $n \neq k$*, then G has finite special rank at most* $\mathbf{r}$*. Moreover,*

$$\mathbf{r}(G) = \mathbf{max}\{\mathbf{r}(G_n) \mid n \in \mathbb{N}\}.$$

The following corollary is immediate.

R.59. COROLLARY. *Let* $\{p_n \mid n \in \mathbb{N}\}$ *be a set of distinct primes. For each* $n \in \mathbb{N}$ *let* G_n *is a* p_n*-group and suppose that* $G = \mathbf{Dr}_{n \in \mathbb{N}} G_n$*. If there is a positive integer* $\mathbf{r}$ *such that* $\mathbf{r}(G_n) \leq \mathbf{r}$ *for each* $n \in \mathbb{N}$*, then G has a finite special rank at most* $\mathbf{r}$*. Moreover,*

$$\mathbf{r}(G) = \mathbf{max}\{\mathbf{r}(G_n) \mid n \in \mathbb{N}\}.$$

If G is a group of special rank 1, then every finitely generated subgroup of G is cyclic. Thus, G is a ***locally cyclic group***. Every locally cyclic group is either periodic or torsion – free, and it is not hard to prove that every torsion – free locally cyclic group is isomorphic to some subgroup of the additive group $\mathbb{Q}$ of rational numbers. The torsion – free locally cyclic groups have been described completely (see, for example, [**KAG1967**, § 30]). On the other hand, if G is a locally cyclic p – group, then G is either a cyclic p – group or the union of the terms of ascending chain of finite cyclic p – groups, and in this latter case, G is a Prüfer p – group. Consequently, if G is a periodic locally cyclic group, then $G = \mathbf{Dr}_{p \in \Pi(G)} G_p$ where G_p is either a cyclic p – group or a Prüfer p – group for each $p \in \Pi(G)$. Thus a periodic group G is locally cyclic if and only if G can be embedded in L = $\mathbf{Dr}_{p \in \mathbf{P}} G_p$ where G_p is a Prüfer p – group for each $p \in \mathbf{P}$, and $\mathbf{P}$ is the set of all primes. Clearly, L is isomorphic to $\mathbf{Q}/\mathbf{Z}$, and hence a group G is locally cyclic if and only if it is isomorphic to some section of $\mathbf{Q}$. These characterizations can be extended to abelian groups of finite special rank **r**.

We say that a group G has ***bounded section rank***, if there is positive integer **b** such that $\mathbf{sr}_p(G) \le \mathbf{b}$ for every prime p. The bounded section rank denoted **bs**(G).

Consider now some relations between special rank, section rank and 0 – rank.

R.60. PROPOSITION. *Let G be a group of finite special rank* ***r****. Then the section p – rank of G is at most* ***r*** *for each prime p. In particular, G has bounded section rank.*

R.61. THEOREM. *Let P be a locally finite p – group for some prime p. Then the following are equivalent:*

(i) P has finite special rank;

(ii) P has finite section p – rank;

(iii) P is a Chernikov group.

Furthermore, in this case, $\mathbf{sr}_p(P) = \mathbf{r}(P)$.

In contrast, for each suitably large prime p ($p > 10^{75}$), A.Yu. Olshanskii [**OA1982**] has constructed infinite 2 – generated groups, all of whose proper subgroups are cyclic of order p. These «Tarski monsters» (so named after A. Tarski, who first predicted their possible existence) clearly are p – groups of special rank 2 and section p – rank 1, and certainly are not Chernikov groups.

Let us dwell a little more detail on abelian groups having finite special rank.

R.62. PROPOSITION. *Let G be an abelian group of finite special rank. Then:*

(i) $\mathbf{r}(G) = \mathbf{r}_0(G) + \mathbf{max}\{\, \mathbf{sr}_p(\mathbf{Tor}(G)) \mid p \in \Pi(G)\,\}$;

(ii) G can be embedded in the direct product of $\mathbf{k}$ *copies of* $\mathbf{Q}/\mathbf{Z}$ *and* $\mathbf{t}$ *copies of* $\mathbf{Q}$, *where* $\mathbf{r} = \mathbf{k} + \mathbf{t}$.

We note also the following useful result.

R.63. PROPOSITION. *Let G be an abelian torsion – free group of finite 0 – rank and let p be a prime such that* $p \notin \mathbf{Sp}(G)$. *Then* $\mathbf{r}(G/G^p) = \mathbf{r}_0(G) = \mathbf{r}(G)$.

The following result concerns with special rank of a polyrational group.

R.64. THEOREM (D.I. Zaitsev [**ZD1971**]). *Let G be a polyrational group. Then* $\mathbf{r}_0(G) = \mathbf{bs}(G) = \mathbf{r}(G)$.

R.65. COROLLARY (M.R. Dixon, L.A. Kurdachenko, N.V. Polyakov [**DKP2007**]). *Let G be a locally generalized radical group of finite 0 –rank* $\mathbf{r}$ *and* $T = \mathbf{Tor}(G)$. *Then* $G/\mathbf{Tor}(G)$ *has finite special rank. Moreover,* $\mathbf{r}_0(G) \leq \mathbf{r}(G/\mathbf{Tor}(G))$ *and* $\mathbf{r}(G/\mathbf{Tor}(G)) \leq \mathbf{r}_0(G) + \mathbf{f}_2(\mathbf{r})$ *for some function* $\mathbf{f}_2$.

Now we consider the structure of locally generalized radical groups of finite special rank. First, we show the following result on locally nilpotent groups, which was the forerunner of **Theorem R.64**.

R.66. THEOREM (V.M. Glushkov [**GV1952[1]**]). *Let G be a locally nilpotent torsion – free group. If G has finite special rank* ***r,*** *then G has finite 0 – rank. Moreover, G is polyrational nilpotent group and* $\mathbf{r}(G) = \mathbf{r}_0(G)$.

As we can see, the case of locally nilpotent groups is turned out to be very specific. In the case of locally finite p – groups, we have the coincidence of special rank with section p – rank. In the case of torsion – free locally nilpotent groups, we have the coincidence of special rank with 0 – rank. Moreover, torsion – free locally nilpotent groups of finite special rank are nilpotent. Naturally, the following question arises: whether a similar result is valid for periodic locally nilpotent groups? The answer on this question is negative. Moreover, there exists a periodic hypercentral group of finite special rank, which is not soluble. The fitting example was constructed by Yu.I. Merzlyakov [**MYu1964**].

Following theorem shows a direct connection between special rank and bounded section rank in locally nilpotent groups.

R.67. THEOREM. *Let G be a locally nilpotent group. Then G has finite special rank if and only if G has bounded section rank. In this case,* $\mathbf{r}(G) = \mathbf{bs}(G)$.

For soluble groups and their generalizations, the connections of all these ranks are quite close. Earlier, we looked at the group $G = A \leftthreetimes \langle x \rangle$ where A is the group of p – adic rational numbers, p is a prime, and $a^x = a^p$ for every element $a \in A$. This group is polyrational and $\mathbf{r}_0(G) = 2$, so that $\mathbf{r}(G) = 2$. But $\mathbf{sr}_p(G) = 1$ and $\mathbf{sr}_q(G) = 2$ for every prime $q \neq p$.

The following theorem describes the structure of locally generalized radical groups of finite special rank.

R.68. THEOREM. *Let G be a locally generalized radical group of finite special rank* ***r****. Then its locally nilpotent radical L is hypercentral and G/L includes a normal abelian subgroup K/L such that G/K is finite. In particular, G is generalized radical, even almost hyperabelian. Moreover,* ***Tor*** *(L) is a direct product of its Chernikov Sylow p –subgroups, L/* ***Tor*** *(L) is nilpotent, K/* ***Tor*** *(L) has finite 0 – rank at most r. In particular, G has finite 0 – rank, moreover,* $\mathbf{r}_0(G) \leq \mathbf{r}$.

R.69. COROLLARY.

(i) Let G be a locally (soluble – by – finite) group of finite special rank. Then, G is almost hyperabelian. (N.S. Chernikov [**CNS1990**]).

(ii) Let G be a locally radical group. If G has finite special rank, then G is a radical group. (B.I. Plotkin [**PB1958**, 16.3.1]).

(iii) Let G be a locally soluble group of finite special rank. Then G has a normal hypercentral subgroup D such that G/D is soluble. In particular, G is hyperabelian. (V.S. Charin [**CVS1957**]).

(iv) Let G be a periodic locally soluble group of finite special rank, and L be a locally nilpotent radical of G. Then L is hypercentral, and G/L includes a normal abelian subgroup K/L

with finite Sylow p – subgroups for all prime p such that G/K is finite. (M.I. Kargapolov [**KMI1959**]).

(v) Let G be a locally generalized radical group of finite special rank. Then G is countable.

Proposition R.60 has the following converse.

R.70. THEOREM. *Let G be a locally generalized radical group. Then G has finite special rank if and only if G has finite bounded section rank.*

Note some corollaries.

R.71. COROLLARY.

(i) Let G be a locally generalized radical group. If every locally radical subgroup of G has finite special rank, then G has finite special rank.

(ii) Let G be a locally (soluble – by – finite) group. If every locally soluble subgroup of G has a finite special rank, then G has a finite special rank (M.R. Dixon, M.J. Evans, H. Smith [**DES1996**]).

Now we show more accurate connections between special rank and bounded section rank. To obtain these connections one needs some important interesting results on finite groups.

R.72. THEOREM (R. Guralnick [**GUR1986**]). *Let G be a finite simple group. Then G can be generated by an involution and a Sylow 2 – subgroup.*

R.73. THEOREM (R. Guralnick [**GUR1989**], A. Lucchini [**LA1989**]). *Let G be a finite group. If each Sylow p – subgroups of G can be generated by **d** elements, then G can be generated by **d** + 1 elements.*

R.74. COROLLARY. *Let G be a finite group. Suppose that there exists a positive integer **b** such that $\mathbf{sr}_p(G) \leq \mathbf{b}$ for each prime p. Then G has a special rank at most **b** + 1.*

Now we can obtain the following specification of **Theorem R.70.**

R.75. THEOREM. *Let G be a locally generalized radical group. If G has finite special rank **r**, then G has finite bounded section rank at most **r**. Conversely, if G has finite bounded section rank **b**, then G has finite special rank at most 3**b** + 1.*

As we have seen, locally generalized radical groups of finite special rank are quite well behaved. However, in general, it appears that the structure of groups of finite special rank can be rather complicated. The best known examples here were constructed by A.Yu. Olshanskii [**OA1982**]. They is called Tarski monsters. These groups can be constructed for each sufficiently large prime p (greater than $10^{75)}$. The groups are 2 – generated infinite simple groups, and each their proper subgroup is generated by a single element, so such a group has special rank 2. In [**OA1991**, Theorem 28.1], the groups have exponent p. Theorem 28.2 of [**OA1991**] provides some further example of a periodic group of rank 2, and [**OA1991**, Theorem 28.3] gives an example of a torsion – free group whose proper subgroups are infinite cyclic. This group also has special rank 2.

The following result of A.Yu. Olshanskii [**OA1991**, Theorem 35.1] allows us to construct further exotic examples of groups of finite special rank.

R.76. THEOREM. *Let* $\{ G_\lambda \mid \lambda \in \Lambda \}$ *be a finite or countable set of non –identity finite or countably infinite groups without involutions. Suppose* $|\Lambda| \geq 2$ *and that n is a sufficiently large odd number (at least* $n \geq 10^{75}$*). Suppose* $G_\lambda \cap G_\mu = \langle 1 \rangle$ *for* $\lambda \neq \mu$, $\lambda, \mu \in \Lambda$*. Then there is a countable simple group* $G = \mathbf{OG}(G_\lambda \mid \lambda \in \Lambda)$*, containing a copy of* G_λ *for all* $\lambda \in \Lambda$ *with the following properties:*

(i) if $x, y \in G$ *and* $x \in G_\lambda \setminus \langle 1 \rangle$, $y \notin G_\lambda$ *for some* $\lambda \in \Lambda$*, then* $G = \langle x,y \rangle$*;*

(ii) every proper subgroup of G is either a cyclic group of order dividing n or is contained in some subgroup conjugate to some G_λ*.*

Hence, A.Yu. Olshanskii groups are 2 – generated and have subgroups which are restricted by the choice of the constituent groups G_λ. An application of this theorem allows us to construct the following examples.

R.77. PROPOSITION.

(i) There is a 2 – generator group G of infinite special rank, all of whose proper subgroups have finite special rank.

(ii) There is a periodic 2 – generator group G of infinite

special rank with all proper subgroups abelian of finite special rank.

(iii) There is an infinitely generated group of infinite special rank with all proper subgroups of finite special rank.

Using the A.Yu. Olshanskii construction, V.N. Obraztsov [**OVN1989**] has constructed the first example of an uncountable group satisfying the minimal condition for all subgroups. It is possible to prove that this group has finite special rank.

R.78. THEOREM. *Let p be a prime such that $p \geq 10^{75}$. Then there exist an uncountable p – group G of finite special rank.*

It seems to be unknown if a torsion – free group of finite special rank could be uncountable.

In conclusion, let us consider a numerical invariant of a group, which has an entirely different nature, but has been very closely associated with the above considerations. This concept is specific, it appeared in different way and from another subject matter. Here we cannot trace any connections to the concept of dimension of the vector space. This concept gives us some «measure of infinity». It has been appeared in the study of groups with weak minimal condition [**ZD1968**], and has been introduced by D.I. Zaitsev. In this paper, D.I. Zaitsev used another term, namely, the «***index of minimality***». However, this term looks unsuitable and not exact. That is why, later on, in his article [**ZK1987**], D.I. Zaitsev proposed another term, the «***minimax rank***». We will use here the term «Zaitsev rank».

Let G be a group and let

$$\langle 1 \rangle = H_0 \leq H_1 \leq \ldots \leq H_{n-1} \leq H_n = G$$

be a finite chain of subgroups of G. Put $\mathscr{C} = \{ H_j \mid 0 \leq j \leq n \}$. Denote by $\mathbf{il}(\mathscr{C})$ the number of the links $H_j \leq H_{j+1}$ such that the index $|H_{j+1}:H_j|$ is infinite.

A group G is said to have ***finite Zaitsev rank*** $\mathbf{r}_z(G) = \mathbf{m}$, if $\mathbf{il}(\mathscr{C}) \leq \mathbf{m}$ for every finite chain of subgroups $\mathscr{C}$ and there exists a chain $\mathscr{D}$ for which this number is exactly $\mathbf{m}$.

Otherwise, we will say that G ***has infinite Zaitsev rank***. Of course, if G is a finite group, then $\mathbf{r}_z(G) = 0$.

Let H, K be the subgroups of a group G and $H \leq K$. We say that a ***link $H \leq K$ is infinite*** if the index $|K:H|$ is infinite, and say that a link $H \leq K$ is ***minimal infinite*** if it is infinite for every subgroup L such that $H \leq L \leq K$ one of the indexes $|L:H|$ or $|K:L|$ is finite.

Suppose that a group G has finite Zaitsev rank and let $\mathscr{D}$ be a finite chain of subgroups such that $\mathbf{il}(\mathscr{D}) = \mathbf{r}_z(G)$. Let $H \leq K$ be the link of this chain such that the index $|K:H|$ is infinite. Let L be a subgroup of G with the property $H < L < K$. Suppose that both indexes $|K:L|$ and $|L:H|$ are infinite and consider the chain $\mathscr{D} \cup \{ L \}$. Clearly this chain is finite, and $\mathbf{il}(\mathscr{D} \cup \{ L \}) = \mathbf{il}(\mathscr{D}) + 1$, which contradicts to the choice of $\mathscr{D}$. This contradiction shows that every link $H \leq K$ of $\mathscr{D}$ with infinite index $|K:H|$ is minimal infinite.

Note the original initial properties of Zaitsev rank.

R.79. PROPOSITION. *Let G be a group, H be a subgroup of G and L be a normal subgroup of G. Then:*

(i) if a group G has finite Zaitsev rank and H has a finite

index, then $\mathbf{r}_z(G) = \mathbf{r}_z(H)$*;*

(ii) if a group G has finite Zaitsev rank, then H has a finite Zaitsev rank, concretely, $\mathbf{r}_z(H) \leq \mathbf{r}_z(G)$*;*

(iii) if a group G has finite Zaitsev rank and L is finite, then $\mathbf{r}_z(G) = \mathbf{r}_z(G/L)$*;*

(iv) if a group G has finite Zaitsev rank, then G/L has finite Zaitsev rank, concretely, $\mathbf{r}_z(G) = \mathbf{r}_z(L) + \mathbf{r}_z(G/L)$*;*

(v) if L and G/L have finite Zaitsev rank, then the entire group G has finite Zaitsev rank.

The groups of finite Zaitsev rank are proved to be very closely related to the following important type of groups.

A group G is called ***minimax***, if G has a finite subnormal series whose factors satisfy either condition **Min** or **Max**.

First, these groups appeared in the paper of R. Baer [**BR1953**]. However, D.J.S. Robinson [**RD1967**] has initiated the first fundamental study of soluble minimax groups. He also introduced the term a «minimax group» [**RD1967**]. In the paper [**BR1968**], R. Baer used the term a «poliminimax group», but later on, all authors began using the term a «minimax group».

The theory of soluble – by – finite minimax groups is well developed now. Many authors from different points of view have studied these groups. The minimax groups evolved in the study of distinct finiteness conditions.

R.80. THEOREM. *Let G be a soluble – by – finite group. Then G has a finite Zaitsev rank if and only if G is minimax.*

The following question appears in this connection: *is Zaitsev rank of every minimax group finite?*

The answer is negative. There exists a periodic uncountable group G satisfying **Min** [**OVN1989**]. This group has an ascending chain

$$<1> = D_0 \leq D_1 \leq \ldots D_\alpha \leq D_{\alpha+1} \leq \ldots D_\gamma = G$$

where $\gamma = \omega_1$ is a first uncountable ordinal. In particular, for each positive integer n, a group G has a finite chain $\mathscr{C}$ such that $\mathbf{il}(\mathscr{C}) = n$. This proves that G has infinite Zaitsev rank.

We remark the following Corollary of **Theorem R.80**.

R.81. COROLLARY. *Let G be a soluble – by – finite group. If G has a finite Zaitsev rank, then G has a finite 0 – rank.*

Let G be an abelian group of finite 0 – rank. We choose in G the maximal $\mathbb{Z}$ – independent subset M and put A = < M >. Then the factor – group G/A is periodic. Denote by **Sp**(G) the set of all primes p such that the Sylow p – subgroup of G/A is infinite. If B is another free abelian subgroup of G such that G/B is periodic, then the both factors $A/(A \cap B)$ and $B/(A \cap B)$ are finite. This shows that the set **Sp**(G) is independent of a choice of a subgroup A, i.e. **Sp**(G) is an invariant of a group G. The set **Sp**(G) is called ***the spectrum*** of a group G. If G includes a normal subgroup H, then it is not hard to see that **Sp**(G) = **Sp**(H) $\cup$ **Sp**(G/H).

Let now G be a soluble group of finite 0 – rank. We define **Sp**(G) as ***an union of the spectrums of the factors of the derived series of G***. It is not hard to see that **Sp**(G) can be defined as a union of the spectrums of the factors of arbitrary series of normal subgroups of G with abelian factors.

We note that a soluble – by – finite minimax group has a finite spectrum. The following property of minimax groups is very useful.

R.82. PROPOSITION. *If A and B are abelian minimax groups then their tensor product $A \otimes B$ is also minimax.*

In particular, it played an important role in the proof of the following assertion.

R.83. PROPOSITION (D.I. Zaitsev [**ZD1971**]). *Let G be a nilpotent group and suppose that G/[G,G] is a minimax group. Then G is also minimax.*

R.84. COROLLARY (D.I. Zaitsev [**ZD1971**]). *Let G be a locally nilpotent torsion – free group of finite 0 – rank. If G/[G,G] is minimax and* ***Sp*** *(G/[G,G]) = π, then G is minimax and* ***Sp*** *(G) = π.*

Now we can apply the mentioned results to finitely generated locally generalized radical groups having finite 0 – rank.

R.85. THEOREM. *Let G be a locally generalized radical group of finite 0 – rank. If G is finitely generated, then G/*__*Tor*__ *(G) is soluble – by –finite and minimax.*

Almost soluble groups of finite Zaitsev rank arise in consideration of different groups with the finiteness conditions, in particular, the weak minimal and weak maximum conditions on various types of subgroups. These conditions, among others, will be discussed in other chapters of this book.

Chapter

4

Normal subgroups and their influence on the group's structure

The presence of a large family of normal subgroups in a group has a very strong, and, sometime, deterministic effect on the group properties and structure. The clearest example of this are abelian groups. They constitute the greater part of the class of Dedekind groups, i.e. the groups, all of subgroups of whose are normal. The theory of abelian groups is one of the most advanced algebraic theories. These considerations lead to the conclusion that the study of groups having «large» in some sense family of normal subgroups can be very fruitful. This is really true, and in this chapter we will show some important (in our opinion) results obtained in the study of groups with a «large» family of normal subgroups.

The first natural step here is consideration of the groups, in which the family of all normal subgroups coincides with a family of all subgroups, or, in other words, the groups all of whose subgroups are normal. As we mentioned, such groups are called ***Dedekind groups***. R. Dedekind in his paper [**DR1897**] has studied finite groups whose subgroups are normal. Much later, in the paper [**BR1933**], R. Baer received a full description of such groups, both finite and infinite. Here is this account.

N.1. THEOREM. *Let G be a group whose subgroups are normal. If G is non – abelian, then* $G = Q \times E \times B$ *where Q is a quaternion group, E is an elementary abelian 2 – subgroup and B is a periodic abelian subgroup such that* $2 \notin \Pi(B)$*.*

If all abelian subgroup of a group G are normal, then every cyclic subgroup of G is normal. It implies the normality of all subgroups of G. The dual situation is much more diverse.

A group G is said to be ***metahamiltonian*** if every non – abelian subgroup of G is normal.

G.M. Romalis and N.F. Sesekin [**RS1966**] initiated the study of these groups. The term a metahamiltonian group also belongs to them. In the articles [**SR1968**, **RS1970**] some properties of metahamiltonian groups were obtained.

It immediately raises the question of whether in general, and under what the restrictions in particular, the study of metahamiltonian groups is realistic task. Thus, mentioned above groups, all of whose proper subgroups are abelian, is obviously metahamiltonian. Very exotic examples of such groups, constructed by A.Yu. Olshanskii, indicate the need for serious restrictions. In other words, it is necessary to find out under what (the widest) restrictions it is possible to determine (more or less detailed) structure of metahamiltonian groups. S.N. Chernikov [**CSN1970**] created one of such efficient restrictions.

A group G is said to be ***locally graded*** if every non – trivial finitely generated subgroup of G includes a proper subgroup of finite index.

N.2. THEOREM (S.N. Chernikov [**CSN1970**]). *Let G be a locally graded metahamiltonian group. Then* $[G,G]$ *is a finite p – subgroup for some prime p.*

In particular, every locally graded metahamiltonian group is soluble. Thus, we come to another natural restriction, with which the description of metahamiltonian groups becomes real – the condition of solvability. The first step is the study of soluble groups, whose proper subgroups are abelian. Finite groups with this property has been described by G.A. Miller and H. Moreno [**MM1903**]. A detailed description of such groups was obtained by L. Redei, and is as follows.

N.3. THEOREM (L. Redei [**RL1947**]). *Let G be a finite non – abelian group whose proper subgroups are abelian. Then:*

(i) if G is a p – group then G is a group of one of the following types:

(ia) G is a quaternion group of order 8;

(ib) $G = \langle a,b \rangle$, and $|a| = p^k$, $|b| = p^t$, $a^b = a^m$ where $m = 1 + p^{k-1}$, $k \geq 2$, $t \geq 1$ and $|G| = p^{k+t}$;

(ic) $G = \langle a,b \rangle$ and $|a| = p^k$, $|b| = p^t$, $[a,b] = c$, $|c| = p$, $|G| = p^{k+t+1}$;

(ii) if G is not a p – group, then $G = AC$ where A is a minimal normal elementary abelian p – subgroup and C is a cyclic q –subgroup where p, q are primes and $p \neq q$.

It is possible to prove that an infinite soluble group, whose proper subgroups are abelian, is abelian, so that the finite case is the main one here.

Significant progress in the study of finite metahamiltonian groups was achieved by A.A. Makhnev [**MAA1976**].

N.4. THEOREM (A.A. Makhnev [**MAA1976**]). *Let G be a finite metahamiltonian p – group where p is a prime. Then:*

*(i) **Frat** $([G,G]) \leq \zeta(G)$;*

(ii) every non – abelian subgroup of G includes $[G,G]$;

(iii) $[G,G]$ is a group of one of the following types:

(iiia) $[G,G]$ is a cyclic subgroup;

(iiib) $[G,G] = C \times D$ *where* C *is a cyclic subgroup,* $|D| = p$;

(iiic) $[G,G] = E_1 \times E_2 \times E_3$ *where* $|E_j| = p$, $1 \leq j \leq 3$;

(iv) if $p = 2$ *and* $[G,G]$ *is elementary abelian, then* $[G,G] \leq \zeta(G)$;

(v) if $p = 2$ *and* $\zeta(G)$ *does not include* $[G,G]$, *then* G *is a metacyclic group.*

Note the following important property of metahamiltonian groups.

N.5. THEOREM (M. de Falco, F. de Giovanni, C. Muzella [**deFdeGM2013**]). *Let* G *be a locally graded metahamiltonian group. If* H *is a non – abelian subgroup of* G, *then* H *includes* $[G,G]$.

A complete description of metahamiltonian groups has been obtained by N.F. Kuzennyi and N.N. Semko in their series of articles [**KS1983**, **KS1985**, **KS1986**, **KS1987**, **KS1989**, **KS1990**, **SK1987**]. We now consider one of the main results of these studies, on which the further description of metahamiltonian groups were based. It describes the derived subgroup of a metahamiltonian group. We present this result taking into account **Theorem N.5.**

N.6. THEOREM (N.F. Kuzennyi, N.N. Semko [**KS1996**, Theorem 2.2.2]). *Let* G *be a non – abelian locally graded metahamiltonian group and* $K = [G,G]$. *Then* K *is a finite* p *– subgroup,* p *is a prime, and one of following holds:*

(i) $|K| = p$;

(ii) K is a cyclic subgroup of order p^d, where $d > 1$, G is a nilpotent group such that **Frat** *(K) $\leq \zeta$ (G);*

(iii) $K = C \times D$ where C is a cyclic p – subgroup, $|C| > p$, $|D| = p$, G is a nilpotent group such that **Frat** *(K) $\leq \zeta$ (G);*

(iv) $K = C \times D$ where $|C| = |D| = p$, G is a nilpotent group and if $p = 2$, then $K \leq \zeta$ (G);

(v) $K = C \times D \times E$ where $|C| = |D| = |E| = p$, G is a nilpotent group and if $p = 2$, then $K \leq \zeta$ (G);

(vi) K is an elementary abelian p – subgroup, $|K| > p$ and a group G is non – nilpotent;

(vii) K is a non – abelian subgroup of order p^3, having exponent p, $p > 3$, G is a periodic non – nilpotent group and K is a Sylow
p –subgroup of G;

(viii) K is a quaternion group, G is a periodic non – nilpotent group and K is a Sylow 2 – subgroup of G.

Subsequent study of metahamiltonian groups was carried out in accordance with the type of derived subgroups. For each of these types, a complete description, in some cases brought to the smallest details, has been obtained. We will not give it here, because it is quite bulky. It can be found in the above mentioned articles of N.F. Kuzennyi and N.N. Semko and in their book [**KS1996**].

It is noteworthy that S.N. Chernikov was the initiator of systematic research on the effect of the system of normal subgroups on the group structure. He writes about it in his articles [**CSN1967**, **CSN1969**, **CSN1971**]. In the paper [**CSN1967**], he considered groups all of whose infinite subgroups are normal.

Naturally, in infinite groups, one could expect that the structure of the infinite subgroups plays a dominant role. This class of groups is wider than the class of Dedekind groups. S.N. Chernikov obtained the following description of it.

N.7. THEOREM (S.N. Chernikov [**CSN1967**]). *Let G be an infinite group all of whose infinite subgroups are normal. Then:*

(i) if G is non – abelian, then G is periodic;

(ii) if G is locally finite then G is either Dedekind, or G includes a normal Prüfer subgroup P such that G/P is a finite Dedekind group.

Note that **Theorem N.7** was verbalized in a different form than it was in [**CSN1967**].

If all cyclic subgroup of a group G are normal, then every subgroup of G is normal. Therefore, it is natural to study the groups whose non – cyclic subgroups are normal. Those groups were studied in the articles of F.N. Liman [**LF1967**, **LF1968**, **LF1968[1]**]. As in the case of metahamiltonian groups, the existing of developed by A.Yu. Olshanskii exotic examples of groups, whose proper subgroups are cyclic, indicate the need for further restrictions. Obviously, the groups, whose non – cyclic subgroups are normal, are metahamiltonian. Therefore, the restriction *to be a locally graded group* could be an appropriate here. **Theorem N.2** shows that this restriction implies the solvability of a group. Therefore, in the results below we will consider locally graded groups, whose non – cyclic subgroups are normal, while in his original works F.N. Liman implied some weaker restrictions.

In consideration of periodic groups, the case of an infinite group is special since every infinite periodic group is non – cyclic. As for the case of finite groups, and for the case of infinite periodic groups, the main case is the case of p – groups where p is a prime. Here the case when p = 2 as usual takes a particular place. It does not require additional restrictions. From the main result of an article of S.P. Strunkov [**SS1967**] it follows that every infinite 2 – group includes an infinite abelian subgroup. It follows that a 2 – group, whose non – cyclic subgroups are normal, is soluble.

N.8. THEOREM (F.N. Liman [**LF1968**]). *Let G be a 2 – group, whose non – cyclic subgroups are normal. Then G is a group of one of following types:*

(i) G is a Dedekind group;

(ii) $G = \langle a \rangle\langle b \rangle$, *where* $a^8 = b^4 = 1$, $a^4 = b^2$, $a^b = a^{-1}$;

(iii) $G = \langle a \rangle\langle b \rangle$, *where* $a^8 = b^8 = 1$, $a^4 = b^4$, $a^b = a^{-1}$;

(iv) $G = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle$, *where* $|a| = 2^n$, $b^2 = c^2 = 1$, $[a,b] = [a,c] = 1$, $[b,c] = a^m$ *where* $m = 2^{n-1}$;

(v) $G = (A \times \langle b \rangle) \leftthreetimes \langle c \rangle$, *where A is a Prüfer 2 – group,* $A \leq \zeta(G)$, $b^2 = c^2 = 1$, $[b,c] = a$ *where* $a \in A$ *and* $a^2 = 1$;

(vi) $G = (Q \times \langle b \rangle) \leftthreetimes \langle c \rangle$, *where* $Q = \langle a \rangle\langle d \rangle$ *is a quaternion group,* $[Q,c] = \langle 1 \rangle$, $b^2 = c^2 = 1$, $[b,c] = a^2$;

(vii) $G = \langle a \rangle \leftthreetimes \langle b \rangle$, *where* $|a| = 2^n$, $|b| = 2^k$, $n \geq 2$, $k \geq 1$, $[a,b] = a^m$ *where* $m = 2^{n-1}$;

(viii) $G = (\langle a \rangle \times \langle b \rangle)\langle c \rangle$, *where* $a^4 = b^4 = c^4 = 1$, $c^2 = a^2b^2$, $[a,c] = a^2$, $[c,b] = c^2$;

(ix) $G = (\langle a \rangle \times \langle b \rangle)\langle c \rangle\langle d \rangle$, *where* $a^4 = b^4 = c^4 = d^4 = 1$, $c^2 = d^2 = a^2b^2$, $[a,c] = a^2$, $[c,b] = c^2$, $[d,a] = [d,c] = d^2$, $[b,d] = b^2$;

(x) $G = Q \times \langle a \rangle$, *where* Q *is a quaternion group,* $|a| = 2^n$, $n \geq 2$;

(xi) $G = Q \times A$, *where* Q *is a quaternion group,* A *is a Prüfer* 2*-group.*

Conversely, in each of these groups every non-cyclic subgroup is normal.

The case of $p \neq 2$ is not so rich.

N.9. THEOREM (F.N. Liman [**LF1967**]). *Let* G *be a locally graded* p*-group,* p *is an odd prime, whose non-cyclic subgroups are normal. Then* G *is a group of one of following types:*

(i) G *is an abelian group;*

(ii) $G = (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle$, *where* $|a| = p^n$, $b^p = c^p = 1$, $[a,b] = [a,c] = 1$, $[b,c] = a^m$ *where* $m = p^{n-1}$;

(iii) $G = (A \times \langle b \rangle) \rtimes \langle c \rangle$, *where* A *is a Prüfer* p*-group,* $A \leq \zeta(G)$, $b^p = c^p = 1$, $[b,c] = a$ *where* $a \in A$ *and* $a^p = 1$;

(iv) $G = \langle a \rangle \rtimes \langle b \rangle$, *where* $|a| = p^n$, $|b| = p^k$, $n \geq 2$, $k \geq 1$, $[a,b] = a^m$ *where* $m = p^{n-1}$;

(v) $G = (\langle a \rangle \times \langle b \rangle) \langle c \rangle$, *where* $a^9 = b^3 = c^9 = 1$, $c^2 = a^2b^2$, $[a,c] = b$, $[c,b] = c^3 = a^{-3}$.

Conversely, in each of these groups every non-cyclic subgroup is normal.

If G is an infinite periodic group, whose non-cyclic subgroups are normal, then an application of **Theorem N.7** allows to obtain the following result.

N.10. THEOREM (F.N. Liman [**LF1967**]). *Let G be an infinite periodic locally graded group and suppose that all its non – cyclic subgroups are normal. Then G is either Dedekind, or* $\zeta(G)$ *includes a Prüfer subgroup P such that G/P is a finite Dedekind group. Moreover, there is a prime p such that every Sylow* p' *– subgroup of G/P is cyclic and Sylow p – subgroup of G/P either is a quaternion group or an elementary abelian p – subgroup of order* p^2.

The case when G is a finite group, whose non – cyclic subgroups are normal, was also described in detail by F.N. Liman [**LF1967**] and A.D. Ustyuzhaninov [**UA1967**].

The class of non – periodic groups, whose non – cyclic subgroups are normal, turned out to be less diverse.

N.11. THEOREM (F.N. Liman [**LF1968**]). *Let G be an infinite non –periodic locally graded group and suppose that all non – cyclic subgroups of G are normal. Then G is a group of one of the following types:*

(i) G is an abelian group;

(ii) $G = Q \times B$ *where Q is a quaternion group, B either is infinite cyclic, or* $B \cong \mathbb{Q}_2$;

(iii) $G = \langle a \rangle \rtimes \langle b \rangle$, *where* $|a| = p^n$ *and* $n \geq 2$ *whenever* $p = 2$, *b has infinite order,* $a^b = a^m$ *where* $m = p^{n-1}$.

Some weaker conditions were considered in the work of M. Herzog, P. Longobardi, M. Maj and A. Mann [**HLMM2000**]. They considered the groups with the following property:

For each element $x \in G$ either a cyclic subgroup $< x >$ is normal in G or a subgroup $< x, x^g >$ is normal for every element $g \notin N_G(< x >)$.

In the paper [**HLMM2000**], some quite clear description of those groups were obtained. It splits on some different cases.

N.12. THEOREM (M. Herzog, P. Longobardi, M. Maj, A. Mann [**HLMM2000**]). *Let G be a non – abelian finite p – group for some prime p. Suppose that for each element $x \in G$ either the subgroup $< x >$ is normal in G or $< x, x^g >$ is normal for every element $g \notin N_G(< x >)$. Then:*

(i) if $p = 2$ then $[G,G]$ has exponent at most 4, if $p > 2$ then $[G,G]$ is an elementary abelian p – subgroup;

*(ii) if $p > 3$, then **ncl** $(G) \leq 2$;*

(iii) if $p = 2$ then $G^4 \leq \zeta(G)$, if $p > 2$ then $G^p \leq \zeta(G)$;

(iv) if $p = 2$ then $|[G,G]| \leq 2^6$, if $p > 2$ then $|[G,G]| \leq p^2$;

*(v) if $p > 2$ and **ncl** $(G) = 3$, then $p = 3$ and $G = AB$ where $A = < a,b >$ and a, b satisfy $a^9 = b^9 = 1$, $a^3 = b^{-3}$, $a^b = ac$, $c = [a,b]$, $c^3 = 1$, $[c,b] = 1$, $[c,a] = a^{-3}$, B has exponent 3, **ncl** $(B) \leq 2$; $[B,B] \leq < a^3 >$, $[B,a] = <1>$, $[B,b] \leq < a^3 >$.*

N.13. THEOREM (M. Herzog, P. Longobardi, M. Maj, A. Mann [**HLMM2000**]). *Let G be a non – abelian finite group. Suppose that for each element $x \in G$ either a subgroup $< x >$ is normal in G or $< x, x^g >$ is normal for every element $g \notin N_G(< x >)$. Then:*

(i) G is soluble;

*(ii) **zl** $(G) \leq 3$;*

(iii) if $H = G/\zeta_\infty(G)$, then $H = A \rtimes \langle d \rangle$, A is an elementary abelian p – subgroup of order at most p^2. Moreover, if $|A| = p^2$, then d has odd order dividing $p + 1$.

The existing of Tarski monster shows that, as in previous cases, the actual study of infinite groups with the above property is possible only under certain additional restrictions.

N.14. THEOREM (M. Herzog, P. Longobardi, M. Maj, A. Mann [**HLMM2000**]). *Let G be a non – abelian infinite locally graded group. Suppose that for each element $x \in G$ either a subgroup $\langle x \rangle$ is normal in G or $\langle x, x^g \rangle$ is normal for every element $g \notin \mathbf{N}_G(\langle x \rangle)$. Then:*

*(i) G is soluble and **dl** $(G) \leq 3$;*

(ii) if G is not periodic, then $|[G,G]| = p$;

(iii) G can not be torsion – free;

(iv) if G is locally finite, then $\zeta_\infty(G) = \zeta_3(G)$ and either $G = \zeta_3(G)$ or $G/\zeta_3(G) = A \rtimes \langle d \rangle$, A is an elementary abelian p – subgroup of order at most p^2. Moreover if $|A| = p^2$, then d has odd order dividing $p + 1$;

*(v) if G is locally nilpotent, then G is nilpotent and **ncl** $(G) \leq 3$, moreover, if G is not periodic, then $|[G,G]| = p$ and $[G,G] \leq \zeta(G)$.*

As we already mentioned, S.N. Chernikov initiated the study of groups G, whose family $\mathfrak{L}_{\mathbf{norm}}(G)$ of all normal subgroup is «very big», or the family $\mathfrak{L}_{\mathbf{non-norm}}(G)$ of all non – normal subgroup is «very small» in some sense. In this regard, we note that, for many types of nilpotent and metabelian groups G the family $\mathfrak{L}_{norm}(G)$ has cardinality $2^{card(G)}$ (see, for example [**BOSW1984**]). However, what does it mean «to be very big» or «to be very small» for infinite groups? There may be a variety of treatments that lead to a number of interesting results. One natural approach here is in consideration of groups whose family $\mathfrak{L}_{non-norm}(G)$ satisfies some finiteness conditions. First finiteness condition was the condition «to be finite», therefore the first task that immediately arises here is to study groups in which the family $\mathfrak{L}_{non-norm}(G)$ is finite. It was done in the article [**HL1990**].

N.15. THEOREM (N.S. Hekster, H.W. Lenstra, Jr. [**HL1990**]). *Let G be an infinite group and suppose the family of all non – normal subgroups of G is finite. Then G is a group of one of following types:*

(i) G is a Dedekind group;

(ii) $G = P \times B$ where P is a p – subgroup for some prime p, $\zeta(P)$ includes a Prüfer subgroup C such that P/C is finite and abelian; B is a finite Dedekind p' – group.

Conversely, if G is a group of type (ii), then the family of all non – normal subgroups of G is finite.

The minimal condition is one of the most useful finiteness conditions. Groups in which the family $\mathfrak{L}_{non-norm}(G)$ satisfies the minimal condition for all subgroups have been studied by S.N. Chernikov in [**CSN1971[1]**], where the following results were obtained.

N.16. THEOREM (S.N. Chernikov [**CSN1971[1]**]). *Let G be an infinite group and suppose that the family* $\mathfrak{L}_{non-norm}(G)$ *satisfies a minimal condition. Then:*

(i) if G is not periodic, then G is abelian;

(ii) if G is locally finite then either G is Dedekind or G is a Chernikov group.

We note that in the articles [**CSN1967**, **CSN1971[1]**] some generalizations of these results have been obtained, but we refrain from discussing these generalizations since they have been described in some surveys (see, [**CSN1969**, **CSN1970**, **CZ1988**, **ZKC1972**]). We should also point out on the paper [**PW1978**] where the given conditions were weakened even further.

The maximal condition is dual to the minimal condition. The groups, in which the family $\mathfrak{L}_{non-norm}(G)$ satisfies the maximal condition, were studied in the articles of L.A. Kurdachenko, N.F. Kuzennyi, N.N. Semko [**KKS1987**] and G. Cutolo [**CG1991**]. We note, that here the situation looks more diverse. The main results of the above mentioned articles are summarized as following.

N.17. THEOREM. *Let G be a locally graded group and suppose that the family* $\mathfrak{L}_{non-norm}(G)$ *satisfies the maximal condition. Then G is a group of one of the following types:*

(i) G is a polycyclic – by – finite group;

(ii) G is a Dedekind group;

(iii) $\zeta(G)$ *includes a Prüfer p – subgroup P such that G/P is a finitely generated Dedekind group;*

(iv) $G = H \times D$ *where* $H \cong \mathbf{Q}_2$ *and D is a finite non – abelian Dedekind group.*

Next, we note that if all finitely generated subgroups of a group G are normal, then all subgroups of G are normal in G. By contrast, groups all of whose infinitely generated subgroups are normal, have more complicated structure. Here, by ***infinitely generated*** we mean not finitely generated. In this case, the family $\mathfrak{L}_{non-norm}(G)$ consists of only finitely generated subgroups. Such groups have been studied in the articles of L.A. Kurdachenko, V.V. Pylaev [**KP1989**], G. Cutolo [**CG1991**] and G. Cutolo, L.A. Kurdachenko [**CK1995**]. The main results of these papers can be formulated in the following way.

N.18. THEOREM. *Let G be a group that has an ascending series of subgroups whose factors are locally (soluble – by – finite). Every infinitely generated subgroup of G is normal if and only if G is a group of one of the following types:*

(i) G is a Dedekind group;

(ii) G includes a normal Prüfer p – subgroup P such that G/P is a finitely generated Dedekind group;

(iii) $G = H \times D$ where $H \cong \mathbf{Q}_2$ and D is a finite non – abelian Dedekind group;

(iv) G satisfies the following conditions:

(iva) $\zeta(G)$ includes a Prüfer p – subgroup P such that G/P is a minimax abelian group with finite periodic part;

(ivb) $\mathbf{Sp}(G/P) = \{p\}$;

(ivc) $G/\mathbf{FC}(G)$ is torsion – free;

(ivd) if A is an abelian subgroup in G, then $A/(A \cap P)$ is finitely generated;

(v) G satisfies the following conditions:

(va) $G = (A \times T) \rtimes \langle g \rangle$, where $A \cong \mathbf{Q}_p$ for some prime p

and T is a finite Dedekind group;

(vb) if T is non – abelian, then p = 2;

(vc) every subgroup of the Sylow p – subgroup T_p of T is < g > – invariant;

(vd) there exists a positive integer k such that $a^g = a^k$ where $k = p^r$ or $k = p^{-r}$, for each $a \in AT_{p'}$ (here $T_{p'}$ is a Sylow p' – subgroup of T).

Now we consider the following interesting generalization of the original minimal and maximal conditions, which are closely linked with the introduced above concept of Zaitsev rank. These conditions were introduced by R. Baer [**BR1968**] and D.I. Zaitsev [**ZD1968**] and were called the weak maximal and weak minimal conditions. These conditions have proved to be very successful, they stimulated a variety of researches in the area. The results of the published researches have been reflected in the survey [**KK1992**]. Observe that the minimax groups have been appearing in the study of groups satisfying the weak minimal and weak maximal conditions, like Chernikov groups in the study of groups with the minimal conditions for distinct subgroups, and polycyclic – by – finite groups in the study of groups with the maximal conditions.

Let G be a group and $\mathfrak{M}$ be a family of some subgroups of G. We say that $\mathfrak{M}$ satisfies ***the weak maximal*** (*respectively* ***minimal***) ***condition*** or G satisfies ***the weak maximal*** (respectively ***minimal***) condition for $\mathfrak{M}$ – subgroups, if for every ascending (respectively descending) series $\{ H_n \mid n \in \mathbf{N} \}$ subgroups of family $\mathfrak{M}$ there exists a number $\mathbf{m} \in \mathbf{N}$ such that the indexes $|H_{n+1}:H_n|$ (respectively $|H_n:H_{n+1}|$) are finite for all $n \geq \mathbf{m}$.

The first results about groups with the weak minimal and maximal conditions for all subgroups have been obtained by D.I. Zaitsev [**ZD1968**, **ZD1971[1]**]. His main result is the following

N.19. THEOREM. *Let G be a locally (soluble – by – finite) group. Then G satisfies the weak minimal (respectively maximal) condition for all subgroups if and only if G is soluble – by – finite and minimax.*

Next natural step is investigation of groups with the weak minimal and weak maximal conditions for abelian subgroups.

N.20. THEOREM (R. Baer [**BR1968**], D.I. Zaitsev [**ZD1969**]). *Let G be a radical group. Then G satisfies the weak minimal (respectively maximal) condition for abelian subgroups if and only if G is a soluble minimax group.*

Many researchers studied groups with the week minimal and maximal conditions for different families $\mathfrak{M}$ of subgroups. From the variety of the results, we present here only those that are directly related to the topic. If $\mathfrak{M} = \mathfrak{L}_{\mathbf{non-norm}}(G)$, then we obtain groups with the weak minimal condition (respectively the weak maximal condition) on non – normal subgroups. The structure of such groups was described by L.A. Kurdachenko and V.E. Goretskij.

N.21. THEOREM (L.A. Kurdachenko, V.E. Goretskij [**KLGV1989**]). *Let G be a locally (soluble – by – finite) group. Then G satisfies the weak minimal (respectively maximal) condition for non – normal subgroups if and only if G is a soluble – by – finite minimax group.*

The groups G for which the family $\mathfrak{L}_{non-norm}$(G) consists of the subgroups of finite special rank only have been considered in the paper [**EK2004**].

N.22. THEOREM (M.J. Evans, Y. Kim [**EK2004**]). *Let G be a locally (soluble – by – finite) group of infinite special rank. If every subgroup of G of infinite special rank is normal, then G is a Dedekind group.*

A large body of research devoted to the study of groups in which the family of normal subgroups satisfies the classical minimal and maximal conditions. The group G is said to satisfy ***the minimal*** (respectively ***maximal***) ***condition on normal subgroups*** or shorter the condition **Min – n** (respectively **Max – n**) if the ordered by inclusion family $\mathfrak{L}_{norm}$(G) of all normal subgroups of G satisfies the minimal (respectively maximal) condition.

The first natural step in the study of groups satisfying **Min – n** was the consideration of the locally nilpotent groups which satisfy this condition.

N.23. THEOREM (I.D. Ado [**AI1946**, **AI1947**], S.N. Chernikov [**CNS1947**]). *Let G be a periodic locally nilpotent group. If G satisfies* ***Min − n****, then G is a Chernikov group.*

In this regard, the question about the coincidence of the conditions **Min** и **Min − n** arose. Almost at once it was answered in the negative.

N.24. THEOREM (V.S. Charin [**CVS1949**]). *There exists a group G satisfying the following conditions:* $G = A \leftthreetimes K$ *where A is infinite elementary abelian p – subgroup, p is a prime, K is a Prüfer q –subgroup, q is a prime and* $p \neq q$*, and A is a minimal normal subgroup of G. In particular, G satisfies* ***Min − n*** *but do not satisfies* ***Min****.*

V.S. Charin has also completed the description of the locally nilpotent groups with **Min − n**.

N.25. THEOREM (V.S. Charin [**CVS1953**]). *Let G be a locally nilpotent group. If G satisfies* ***Min − n****, then G is a Chernikov group.*

This result has been extend in a following way.

N.26. THEOREM (D.H. McLain [**McL1959**]). *Let G be a locally supersoluble group. If G satisfies* ***Min − n****, then G is a Chernikov group.*

One of the following steps in the investigation of the groups satisfying **Min − n** was the search on the group classes in which the conditions **Min** – **n** and **Min** coincide. In this way, there were interesting classes of groups, among which the following very broad generalization of hypercentral groups.

Let G be a group and S its G – invariant subset (that is $g^x \in S$ for every $g \in S$ and every $x \in G$). Then $C_G(S)$ is a normal subgroup of G.

The factor – group $G/C_G(S) = \mathbf{Coc}_G(S)$ is called the ***cocentralizer of the set S in the group G***.

The group $\mathbf{Coc}_G(S)$ is isomorphic to some subgroup of $\mathbf{Aut}(< S >^G)$. The influence of cocentralizers of many objects related to the group on the structure of this group is a subject of study in many branches of group theory. Thus, finite group theory has developed many examples where cocentralizers of chief factors play a significant role. Particularly, we can see it in formations theory (note that local formations are defined by restrictions on cocentralizers of the chief factors). Some classes of infinite groups, which arise by restriction on cocentralizers of conjugacy classes, are also studied.

Let $\mathfrak{X}$ be a class of groups. A group G is said ***to have*** $\mathfrak{X}$ ***– conjugacy classes*** (shortly, G is an $\mathfrak{X}$***C – group***) if $\mathbf{Coc}_G(g^G) \in \mathfrak{X}$ for each element $g \in G$.

If $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then the class of $\mathfrak{I}$**C** – groups is exactly the class $\mathfrak{A}$ of all abelian groups. Therefore, class of $\mathfrak{X}$**C** – groups is an extension of the class of abelian groups for every class $\mathfrak{X}$.

If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then we obtain the well – known class of ***FC – groups***. The class of **FC** – groups is an extension of the class $\mathfrak{F}$ of all finite groups and of the class $\mathfrak{A}$ of all abelian groups as well. Consequently, it inhered many properties of both of these classes. Currently, the theory of **FC –** groups is one of best – developed branches of infinite groups theory.

If $\mathfrak{X} = \mathfrak{C}$ is the class of all Chernikov groups, then we obtain ***the class of groups with Chernikov conjugacy classes*** or ***CC – groups***. Ya.D. Polovitsky in [**PYa1964**] introduced this class. The class of **CC –** groups are not investigated as far as **FC** – groups. The study of $\mathfrak{X}$**C** – groups for other important classes $\mathfrak{X}$ has only recently begun.

Let G be a group, $\mathfrak{X}$ a class of groups. Put

$$\mathfrak{X}\mathbf{C}(G) = \{ x \in G \mid \mathbf{Coc}_G(x^G) \in \mathfrak{X} \}.$$

A class of groups $\mathfrak{X}$ is called a ***formation*** if it satisfies the following conditions:

(F1) if $G \in \mathfrak{X}$, H is a normal subgroup of G, then $G/H \in \mathfrak{X}$;

(F2) if H, L are the normal subgroups of a group G such that $G/H, G/L \in \mathfrak{X}$, then $G/(H \cap L) \in \mathfrak{X}$.

We note that if G is a group and $\mathfrak{X}$ is a formation of groups, then $\mathfrak{X}\mathbf{C}(G)$ is a characteristic subgroup of G. In this case, the subgroup $\mathfrak{X}\mathbf{C}(G)$ is called the $\mathfrak{X}$***C – center of the group G.***

A group G is an $\mathfrak{X}$**C *– group*** if and only if $G = \mathfrak{X}\mathbf{C}(G)$.

Starting from $\mathfrak{X}\mathbf{C}$ – center we can construct ***the upper $\mathfrak{X}$C – central series*** of a group G:

$$\langle 1\rangle = C_0 \le C_1 \le \ldots C_\alpha \le C_{\alpha+1} \le \ldots C_\gamma$$

where $C_1 = \mathfrak{X}\mathbf{C}(G)$, $C_{\alpha+1}/C_\alpha = \mathfrak{X}\mathbf{C}(G/C_\alpha)$ for every ordinal $\alpha < \gamma$, $C_\lambda = \bigcup_{\mu<\lambda} C_\mu$ for every limit ordinal λ, and $\mathfrak{X}\mathbf{C}(G/C_\gamma) = \langle 1\rangle$. The last term C_γ of this series is called the ***upper $\mathfrak{X}$C – hypercenter*** and denote by $\mathfrak{X}\mathbf{C}_\infty(G)$.

If $\mathfrak{X}\mathbf{C}_\infty(G) = G$, then the group G is called ***$\mathfrak{X}$C – hypercentral***. If γ is finite, then G is called ***$\mathfrak{X}$C – nilpotent.***

If $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then we obtain the hypercentral groups. If $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then we obtain the ***FC – hypercentral groups***. In this case, we will say about ***the upper FC – central series*** and will use the designation $\mathbf{FC}_\alpha(G)$ for the terms of this series, so $\mathbf{FC}(G) = \mathbf{FC}_1(G)$ is the ***FC – center*** of G, $\mathbf{FC}_\infty(G)$ is the ***upper FC – hypercenter*** of a group G.

A hyperfinite group is an example of an **FC –** hypercentral group. And conversely, a periodic **FC –** hypercentral group is hyperfinite. We also noted that a finitely generated **FC** – hypercentral group is nilpotent – by – finite.

N.27. THEOREM (A.M. Duguid, D.H. McLain [**DMcL1956**]). *Let G be an **FC –** hypercentral group. If G satisfies **Min – n**, then G is a Chernikov group.*

We remark that an **FC –** hypercentral group has a series of normal subgroups, whose factors either are abelian and finitely generated or finite. In this connection, we show the following result.

N.28. THEOREM (V.S. Charin [**CVS1953**]). *Let G be a group having an ascending series of normal subgroups whose factors are abelian groups of finite special rank. If G satisfies* ***Min – n****, then G is a Chernikov group.*

The following result plays a significant role in the study of groups satisfying **Min – n**.

N.29. THEOREM (J.S. Wilson [**WJ1970**]). *Let G be a group and H be a normal subgroup having finite index. If K is a normal subgroup of G, satisfying the minimal (respectively maximal) condition for G –invariant subgroups, then K satisfies the minimal (respectively maximal) condition for H – invariant subgroups.*

This theorem allows reduction to the $\mathfrak{F}$ – perfect subgroups in groups with **Min – n**.

N.30. COROLLARY. *Let G be a group satisfying* ***Min – n****. Then G includes an* $\mathfrak{F}$ *– perfect normal subgroup of finite index satisfying* ***Min – n****.*

The next natural step was the study of soluble groups satisfying **Min – n**.

N.31. THEOREM (R. Baer [**BR1964**]). *Let G be a soluble group satisfying **Min – n**. Then G is periodic.*

The structure of metabelian and metanilpotent groups satisfying **Min – n** has been investigated in [**McD1970**, **SH1974**, **SH1975**]. More precisely, it has been studied the structure of $\mathfrak{F}$ – perfect metanilpotent groups with **Min – n**, since, as **Theorem N.29** shows, the study of groups satisfying **Min – n** is possible with the respect to the subgroup of finite index. We show now the main results of these papers.

N.32. THEOREM (H.L. Silcock [**SH1974**]). *Let G be an $\mathfrak{F}$ – perfect metanilpotent group satisfying **Min – n**. Then the following assertions hold:*

(i) [G,G] is nilpotent;

(ii) if p is a prime and P is the Sylow p – subgroup of G, then P = DS where D is the Sylow p – subgroup of [G,G], S is abelian and [D,S] = <1>. In particular, every Sylow p – subgroup of G is nilpotent;

(iii) G = [G,G] A where A is a divisible Chernikov subgroup and ζ (G) includes the intersection [G,G] $\cap$ A;

(iv) G is countable.

N.33. THEOREM (H.L. Silcock [**SH1974**]). *Let k be a positive integer. Then there exists a soluble group satisfying **Min – n** such that **ncl** ([G,G]) = **k** and **dl** (G) ≤ $\mathbf{log_2 k}$ + 2.*

N.34. COROLLARY (D. McDougall [**McD1970**]). *Let G be an $\mathfrak{F}$ – perfect metabelian group satisfying* ***Min – n****. Then the following assertions hold:*

(i) the Sylow p – subgroups of G are abelian;

(ii) for every set π of primes all Sylow π – subgroups of G are conjugate;

(iii) G = [G,G] ⋋ A where A is a divisible Chernikov subgroup and every complement to [G,G] in G is conjugated with A.

Based on **Corollary N.34**, B. Hartley and D. McDougall [**HMcD1971**] obtained the description of metabelian group G satisfying **Min – n**. More precisely, we can consider an abelian normal subgroup M = [G,G] as an artinian $\mathbb{Z}$A – module over a Chernikov abelian group A. B. Hartley and D. McDougall [**HMcD1971**] obtained the description of M, and therefore obtained a description of the structure of G (look for details the book [**KOS2007**, Chapter 12]).

The case of the soluble groups G satisfying **Min – n** with **dl**(G) ≥ 3 turns out much more complicated. The following result justifies this.

N.35. THEOREM (B. Hartley [**HB1977**]). *Let G be a Charin group (group from* ***Theorem N.24****).*

(i) There exists an uncountable artinian $\mathbb{F}_r$G – module, where r is a prime such that r ∉ Π (G).

(ii) There exists an uncountable soluble group V satisfying ***Min – n*** *such that* ***dl*** *(V) = 3.*

We note one more result on cardinality of soluble groups satisfying **Min – n**.

N.36. THEOREM (B. Hartley [**HB1977**]). *Let G be a soluble group satisfying* ***Min – n****. If* ***dl*** *(G) =* ***k*** *≥ 2, then G has a cardinality at most* $\aleph_{k-2}$.

Here $\aleph_j$ denotes $(j + 1)^{th}$ infinite cardinal.

We note also that **Theorem N.31** cannot be extended to locally soluble groups.

N.37. THEOREM (H. Heineken, J.S. Wilson [**HW1974**]). *There exists a non – periodic locally soluble group satisfying* ***Min – n.***

Now consider a dual condition, the condition **Max – n**. For locally nilpotent groups we have a situation corresponding to the similar case in **Min – n**.

N.38. THEOREM (V.M. Glushkov [**GV1952[1]**]). *Let G be a locally nilpotent group. If G satisfies* ***Max – n****, then G is a finitely generated nilpotent group.*

This result also has been extended in the following way.

N.39. THEOREM (D.H. McLain [**McL1959**]). *Let G be a locally supersoluble group. If G satisfies* ***Max – n****, then G is a finitely generated supersoluble group.*

N.40. THEOREM (A.M. Duguid, D.H. McLain [**DMcL1956**]). *Let G be an* ***FC –*** *hypercentral group. If G satisfies* ***Max – n****, then G is a finitely generated nilpotent – by – finite group.*

Consider the soluble groups satisfying **Max – n**. Here the key role played the article of Ph. Hall [**HP1954**]. First its result is the following.

N.41. THEOREM (Ph. Hall [**HP1954**]). *Let G be a soluble group. If G satisfies **Max – n**, then G is finitely generated.*

It is clear that not every finitely generated soluble group satisfies **Max – n**. Ph. Hall shows some of such groups.

N.42. THEOREM (Ph. Hall [**HP1954**]).

*(i) Every finitely generated abelian – by – (almost polycyclic) group satisfies **Max – n**.*

*(ii) Let H be a polycyclic – by – finite group and A be a group satisfying **Max – n**. Then the wreath product A **wr** H satisfies **Max – n**.*

*(iii) Let K, L be the groups satisfying **Max – n**. If H has no non –trivial central factor, then the wreath product A **wr** H satisfies **Max – n**.*

N.43. COROLLARY (Ph. Hall [**HP1954**]). *There exists only countably many non – isomorphic finitely generated abelian – by – (almost polycyclic) groups.*

However,

N.44. THEOREM (Ph. Hall [**HP1954**]). *Let A be a non – trivial countable abelian group. Then there exists uncountable many non –isomorphic 2 – generator groups D such that $\zeta(D) \cong A$, $\zeta(D/\zeta(D))$ is trivial and $D/\zeta(D)$ is metabelian.*

The next result of Ph. Hall shows that not every finitely generated soluble group satisfies **Max – n**.

N.45. THEOREM (Ph. Hall [**HP1954**]). *There exists a group G satisfying the following conditions:*

(i) ζ (G) is a free abelian subgroup of countable 0 – rank;

(ii) ζ (G/ζ (G)) is trivial;

(iii) G is 2 – generator group;

(iv) G/ζ (G) is metabelian.

We have already had to deal with generalizations of the usual minimal and maximal conditions. It is natural to consider them in relation to the system of normal subgroups. As for the usual minimal conditions the most natural first step here is the study locally nilpotent groups with this conditions. As we have already seen above in **Theorems N.25** and **N.38,** for the locally nilpotent groups the condition **Min – n** implies **Min**, and the condition **Max – n** implies **Max**. Therefore, the question of whether to preserve similar conditions for the weak minimal and maximal normal subgroups raises at once. A negative answer to this question is given by the following example. Let p be a prime and A = $\mathbf{Dr}_{n \in \mathbb{N}} \langle a_n \rangle$ be an elementary abelian p – group. It is not hard to check that A has an automorphism ϕ defined by the rule:

$$\phi(a_1) = a_1,\ \phi(a_{n+1}) = a_{n+1}a_n \text{ for } n > 0.$$

Let G = A $\rtimes \langle \phi \rangle$ be a natural semidirect product of A and its automorphisms group $\langle \phi \rangle$. It is not hard to prove that every proper G – invariant subgroup of A is finite. Since G/A is an infinite cyclic group, G satisfies the weak maximal and weak minimal conditions for normal subgroups. A group G is hypercentral, more precisely

$$\langle 1 \rangle \le \langle a_1 \rangle \le \langle a_1, a_2 \rangle \le \ldots \le \langle a_1, \ldots, a_n \rangle \le \ldots A \le G$$

is the upper central series of G.

Thus, we can see that the case of the weak minimal and maximal conditions even in locally nilpotent groups is much more diverse. Nevertheless,

N.46. THEOREM (L.A. Kurdachenko [**KLA1979**]). *Let G be a locally nilpotent group satisfying the weak minimal (respectively maximal) condition for normal subgroups. Then the following assertions hold:*

(i) if G is a periodic group, then G is a Chernikov group;

(ii) if G is a torsion – free group, then G is minimax;

(iii) if G is a residually finite group, then G is minimax.

We note that in any group, the subgroup generated by the arbitrary family of $\mathfrak{F}$ – perfect subgroups is $\mathfrak{F}$ – perfect. It follows that the subgroup $\mathfrak{F}$(G) generated by all $\mathfrak{F}$ – perfect subgroups is the greatest $\mathfrak{F}$ – perfect subgroup of G. Clearly, this subgroup is characteristic.

Some details of the structure of locally nilpotent groups satisfying the weak minimal condition for normal subgroups discussed in [**KLA1984**, **KLA1990**]. Note some its results.

N.47. THEOREM (L.A. Kurdachenko [**KLA1984**]). *Let G be a locally nilpotent group satisfying the weak minimal condition for normal subgroups. Then the following assertions hold:*

(i) ***Tor*** *(G) satisfies minimal condition for G – invariant subgroups;*

(ii) $\mathfrak{F}$*(G) is a periodic divisible abelian subgroup;*

(iii) G is hypercentral and soluble.

N.48. COROLLARY (L.A. Kurdachenko [**KLA1984**]). *Let G be a locally nilpotent group. Then G satisfies the weak minimal condition for normal subgroups if and only if Tor (G) satisfies minimal condition for G – invariant subgroups and G/* ***Tor*** *(G) is minimax.*

N.49. THEOREM (L.A. Kurdachenko [**KLA1984**]). *Let G be a locally nilpotent group satisfying the weak minimal condition for normal subgroups. If H is a normal subgroup of G such that a factor – group G/H is periodic, then H satisfies the weak minimal condition for normal subgroups.*

N.50. THEOREM (L.A. Kurdachenko [**KLA1984**]). *Let G be a locally nilpotent group satisfying the weak minimal condition for normal subgroups. Then G is countable.*

The structure of hypercentral groups satisfying the weak maximal condition for normal subgroups was clarified in details.

N.51. THEOREM (L.A. Kurdachenko [**KLA1985**]). *Let G be a locally nilpotent group satisfying the weak maximal condition for normal subgroups. Then:*

(i) G is hypercentral if and only if G is soluble;

(ii) $\mathfrak{F}$(G) is a Chernikov group.

N.52. THEOREM (L.A. Kurdachenko [**KLA1985**]). *Let G be a hypercentral group satisfying the weak maximal condition for normal subgroups. Then the following assertions hold:*

(i) G satisfies the weak minimal condition for normal subgroups;

*(ii) **Tor** (G)/$\mathfrak{F}$(G) is nilpotent and has finite exponent;*

*(iii) G = **Tor** (G) A for some minimax subgroup A.*

The study of the weak minimal and weak maximal conditions in some extensions of the class of hypercentral groups was continued in the paper [**KLA1990[1]**]. Next natural step was consideration of the weak minimal and weak maximal conditions in the class of metabelian groups.

Let G be a group and suppose that G includes an abelian normal subgroup A. If H = G/A, then H acts on A by $ah = a^g$, where $h = gA \in H$ and $a \in A$ and this action makes A into a $\mathbb{Z}$H – module. If A is periodic, then very often we may replace A by one of its primary components to assume further that A is a p – subgroup, where p is a prime, so that we come to a p – module over the ring $\mathbb{Z}$H. In this case, the structure of the lower layer $P_1 = \mathbf{\Omega}_1(A)$ of A is significantly influential on the structure of A. Since P_1 is an elementary abelian p – subgroup, we may think of P_1 as a module over the ring $\mathbb{F}_p H$, where $\mathbb{F}_p$ is a prime field of order p. The above approach allows use of both module and ring theoretical methods towards characterization of the groups considered.

In particular, if G is a metabelian group, then its derived subgroup D = [G,G] is an abelian normal subgroup and the factor – group H = G/D is also abelian, so in this case a group ring $\mathbb{Z}H$ is commutative. This explains the fact why the theory of metabelian groups has been very well developed. If G is a metabelian group satisfying the weak minimal or weak maximal condition, then the factor group H = G/[G,G] is abelian and minimax, so that the determining moment in the describing of the structure of G is in clarifying of the structure of a $\mathbb{Z}H$ – module D = [G,G]. The corresponding study was initiated in the article of D.I. Zaitsev, L.A. Kurdachenko and A.V. Tushev [**ZKT1985**]. The structure of these modules has been studied in sufficient detail in the work [**KMK1988**] and [**KLA1995**]. We will not go into the details of this description, because it requires the use of a certain array of very specific concepts. We confine ourselves to some of the results on metabelian groups with the weak minimal conditions.

N.53. THEOREM (D.I. Zaitsev, L.A. Kurdachenko, A.V. Tushev [**ZKT1985**]). *Let G be a metabelian group satisfying the weak minimal condition for normal subgroups. Then G is countable.*

N.54. THEOREM (D.I. Zaitsev, L.A. Kurdachenko, A.V. Tushev [**ZKT1985**]). *Let G be a torsion – free metabelian group satisfying the weak minimal condition for normal subgroups. Then G is minimax.*

N.55. THEOREM (M. Karbe, L.A. Kurdachenko [**KMK1988**]). *Let G be a hyperabelian group satisfying the weak minimal condition for normal subgroups. If G is residually finite, then G is minimax.*

For periodic soluble groups the situation is the same as for the periodic locally nilpotent groups.

N.56. THEOREM (D.I. Zaitsev, L.A. Kurdachenko, A.V. Tushev [**ZKT1985**], M. Karbe [**KaM1987**]). *Let G be a periodic metabelian group satisfying the weak minimal condition for normal subgroups. Then G satisfies the minimal condition for normal subgroups.*

In conclusion, let us turn to yet another aspect of normality. Unlike many other subgroup properties, such for example as «to be a subgroup», «to be an ascendant subgroup», «to be a subnormal subgroup», the property «to be a normal subgroup» is not transitive. Therefore, the question about the groups in which such transitivity holds naturally arises.

A group G is said to be a ***T – group***, if the condition «to be normal subgroup» is transitive, i.e. the facts that a subgroup H is normal in K, and K is normal in G implies that H is normal in G.

In other words, a group G is a **T** – group, if every subnormal subgroups of G is normal. It follows that every nilpotent **T** – group is a Dedekind group.

It is an immediate consequence of the definition that normal subgroups and the factor – groups of **T** – groups are also **T** –groups. However, in general, the same cannot be said about subgroups of **T** – groups. For example, every simple group is a **T** –group but not every subgroup of a simple group is a **T** – group. These speculations naturally lead to the following concept.

Let G be a group. We say that a subgroup H of a group G is ***transitively normal in G*** if H is normal in every subgroup K ≥ H in which H is subnormal (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2006**]).

In [**MV1999**], these subgroups have been introduced under a different name. Namely, a subgroup H of a group G is said to satisfy the ***subnormalizer condition in G*** if for every subgroup K such that H is normal in K we have $\mathbf{N}_G(K) \leq \mathbf{N}_G(H)$.

If G is a group, every subgroup of which is transitively normal, then every subgroup of G is a **T** – group. For such groups there is a special notation.

A group G is said to be a $\overline{\boldsymbol{T}}$ – ***group***, if every subgroup of G is a **T –** group.

Naturally, the condition of «every subgroup of a group is transitively normal» is more strict than the condition «a group is a **T** – group». Therefore it is natural that in the study of $\overline{\mathbf{T}}$ – groups the progress has been more significant than in study of the **T** –groups. In particular it turns out that every finite soluble **T –**group is a $\overline{\mathbf{T}}$ – group. It is coming from the following description of the structure of finite soluble **T** – group, which has been obtained by W. Gaschütz [**GW1957**].

N.57. THEOREM (W. Gaschütz [**GW1957**]). *Let G be a finite soluble* ***T*** *– group, L be a nilpotent residual of G. Then the following assertions hold:*

(i) $G = L \rtimes D$ *where D is a Dedekind group;*

(ii) L is abelian;

(iii) $2 \notin \boldsymbol{\Pi}(L)$*;*

(iv) $\boldsymbol{\Pi}(L) \cap \boldsymbol{\Pi}(G/L) = \varnothing$*;*

(v) every subgroup of L is G – invariant.

The description of infinite soluble $\overline{\mathbf{T}}$ – groups has been obtained by D.J.S. Robinson [**RD1964**]. It turns out to be very similar to the description obtained by W. Gaschütz. A significant difference is in the case of infinite soluble $\overline{\mathbf{T}}$ – groups, where the locally nilpotent residual could be not complemented in G. In the paper [**GYu1962**], the well – known sophisticated construction has been developed. This construction, in particular, allows us to develop examples of periodic groups that are non – splitting extensions of its abelian Hall derived subgroup by an uncountable elementary abelian 2 – group.

N.58. THEOREM (D.J.S. Robinson [**RD1964**]). *Let G be a soluble* $\overline{\boldsymbol{T}}$ *–group, L be a locally nilpotent residual of G. If G is periodic, then the following assertions hold:*

(i) G/L is a Dedekind group;

(ii) L is abelian;

(iii) $2 \notin \boldsymbol{\Pi}(L)$;

(iv) $\boldsymbol{\Pi}(L) \cap \boldsymbol{\Pi}(G/L) = \varnothing$;

(v) every subgroup of L is G – invariant.
If G is non – periodic, then G is abelian.

As we mentioned above, the structure of soluble **T** – groups is much more sophisticated. A quite observing picture of the structure of soluble **T** – groups has been developed by D.J.S. Robinson in his paper [**RD1964**]. We will not come into the details of this description, we note only some general results.

N.59. THEOREM (D.J.S. Robinson [**RD1964**]). *Let G be a soluble* ***T*** *–group. Then the following assertions hold:*

(i) G is metabelian;

(ii) G is locally supersoluble;

(iii) if G is finitely generated and infinite, then G is abelian.

Chapter

5

On interposition of subgroups and its influence on their properties

Let G be a group. For every its subgroup H there exist some naturally defined chains. First of all, this is the following ascending series

$$<1> = H_0 \leq H_1 \leq \ldots H_\alpha \leq H_{\alpha+1} \leq \ldots H_\gamma \leq H_{\gamma+1} = G$$

where $H_1 = H$, $H_2 = \mathbf{N}_G(H_1) = \mathbf{N}_G(H)$, $H_{\alpha+1} = \mathbf{N}_G(H_\alpha)$ for every ordinal $\alpha < \gamma$, $H_\lambda = \bigcup_{\mu<\lambda} H_\mu$ for every limit ordinal λ and $\mathbf{N}_G(H_\gamma) = H_\gamma$. This chain is the ***upper normalizer chain of H in G***. Here the two natural types of subgroups appear. If $H_\gamma = G$, then a subgroup H is ***ascendant in G***. If $H_\gamma = H$ (that is $\mathbf{N}_G(H) = H$), then a subgroup H is called ***self – normalizing in G***. So, we can see that every subgroup is naturally associated with two types of subgroups that are ascendant and self – normalizing subgroups. In this connection, note that the groups, whose subgroups are either ascendant or self – normalizing, have been considered by L.A. Kurdachenko, J. Otal, A. Russo and G. Vincenzi in the paper [**KORV2011**]. Here are some results of this article.

P.1. THEOREM (L.A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [**KORV2011**]). *Let G be a locally finite group, whose subgroups are either ascendant or self – normalizing. If G is not locally nilpotent, then $G = A \leftthreetimes P$ where $P = < x >$ is cyclic p – subgroup for some prime p, $A = [G,G]$ is a normal nilpotent p' – subgroup, $\mathbf{C}_P(A) = < x^p >$, $\mathbf{C}_G(P) = P$. Conversely, if G is a group having this structure, then every subgroup of G is either ascendant or self – normalizing.*

P.2. THEOREM (L.A. Kurdachenko, J. Otal, A. Russo, G. Vincenzi [**KORV2011**]). *Let G be a hyperabelian group, whose subgroups are either ascendant or self – normalizing. Then:*

(i) if G is locally nilpotent, then every subgroup of G is ascendant;

(ii) if G is not periodic, then every subgroup of G is ascendant.

Consider again the upper normalizer chain of a subgroup H. If K is a subgroup of G such that $H_\gamma \leq K$, then H_γ is self – normalizing in K. However, the subgroup K itself could not be self – normalizing in G. Therefore, we come to the following important type of subgroups.

Let G be a group. A subgroup H is called ***weakly abnormal in G*** if every subgroup, containing H (and H itself) is self – normalizing in G.

Using the concept of the normal closure, the following characteristic of the weakly abnormal subgroups was obtained.

P.3. THEOREM (M.S. Ba, Z.I. Borevich [**BB1988**]). *Let G be a group and H a subgroup of G. Then H is weakly abnormal in G if and only if $x \in H^{<x>}$ for each element $x \in G$.*

If H is a maximal subgroup of G such that H is not normal, then $H = N_G(H)$. Thus if $x \notin H$, then $G = \langle H, H^x \rangle$. In particular, $x \in \langle H, H^x \rangle$ and we come to the following type of subgroups.

Let G be a group. A subgroup H is called ***abnormal in G*** if $g \in \langle H, H^g \rangle$ for each element g of G.

The abnormal subgroups have appeared in the paper [**HP1937**] due to Ph. Hall, the term «an abnormal subgroup» also belongs to Ph. Hall, even though it appeared first in the article of R.W. Carter [**CR1961**].

Since $H^{<x>}$ includes $< H, H^x >$, every abnormal subgroup is weakly abnormal, but the converse statement is not true. A corresponding counterexample one can find in [**BB1988**].

By their nature, abnormality is an antagonist to normality: a subgroup H of a group G is *both normal and abnormal only if it coincides with the whole group*. As we saw earlier, every maximal non – normal subgroup is abnormal. More interesting is an example of J. Tits: a subgroup **T**(n,F) of all triangular matrices over field F is abnormal in a general linear group **GL**(n,F) over a field F. Finite (soluble) groups provide us with many assorted examples of abnormal subgroups (see, for examples, the books [**BEA2010**, **DH1992**]). Among them, we note such an important example as the ***Carter subgroups*** (that is a nilpotent self – normalizing subgroup), introduced by R.W. Carter in [**CR1961**]. In this paper R.W. Carter also obtained the following characterization of abnormal subgroups.

P.4. THEOREM (R.W. Carter [**CR1961**]). *Let G be a group and H be a subgroup of G. Then H is abnormal in G if and only if the following two conditions hold:*

(i) if K is a subgroup including H, then K is self – normalizing in G;

(ii) if K, L are two conjugate subgroups including H, then K = L.

In the case of soluble groups, the condition (ii) in **Theorem P.4** could be omitted. For finite groups, this fact was noted in the book of B. Huppert [**HB1967**, p. 733, Theorem 11.17]. For infinite groups, the most general expansion of this result was obtained in [**KKSu2011**, Theorem 1.2].

P.5. THEOREM (V.V. Kirichenko, L.A. Kurdachenko, I.Ya. Subbotin [**KKSu2011**]). *Let G be a hyper – Ñ – group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, including H, is self – normalizing. In other words, in hyper – Ñ –group every weakly abnormal subgroup is abnormal.*

P.6. COROLLARY (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2005**]). *Let G be a radical group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, including H, is self – normalizing. In other words, in radical group every weakly abnormal subgroup is abnormal.*

P.7. COROLLARY (F. de Giovanni, G. Vincenzi [**deGV2001**]). *Let G be a hyperabelian group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, including H, is self – normalizing. In other words, in hyperabelian group every weakly abnormal subgroup is abnormal.*

P.8. COROLLARY. *Let G be a soluble group and H be a subgroup of G. Then H is abnormal in G if and only if every subgroup, including H, is self – normalizing. In other words, in soluble group every weakly abnormal subgroup is abnormal.*

Now we consider another connected to a subgroup H of group G natural series, which is in some sense dual to the upper normalizer chain. This is a descending series

$$G = H_0 \geq H^G = H_1 \geq \ldots H_\alpha \geq H_{\alpha+1} \geq \ldots H_\gamma,$$

defined by the following rule: $H_{\alpha+1} = H^{H_\alpha}$ for every ordinal $\alpha < \gamma$, and $H_\lambda = \bigcap_{\mu<\lambda} H_\mu$ for every limit ordinal λ. This series is called ***the normal closure series of H in G***. The last term H_γ of this series is called ***the lower normal closure of H in G*** and will be denoted by $H^{G,\infty}$. Here again we can distinguish two natural types of subgroups. If $H = H^{G,\infty}$, then a subgroup H is called ***descendant in G.*** An important particular case of descendant subgroups are subnormal subgroups. A subnormal subgroup is exactly a descending subgroup having finite normal closure series. These subgroups strongly affect the structure of a group. For example, it is possible to prove that if every subgroup of a locally (soluble – by – finite) group is descendant, then this group is locally nilpotent. If every subgroup of a group G is subnormal, then, by a remarkable result due to W. Möhres [**M1990**], G is soluble. Subnormal subgroups have been studied very thoroughly for quite a long period of time. We will not delve into this subject, since it is fairly reflected in the book of J.C. Lennox and S.E. Stonehewer [**LS1987**] and the survey of C. Casolo [**CC2008**]. Another extreme case leads us to the following type of subgroups. If $G = H^G$, then a subgroup H is called ***contranormal in G***.

The term a contranormal subgroup has been introduced by J.S. Rose in [**RJ1968**]. Note at once that every subgroup is contranormal in its lower normal closure.

If K is a subgroup of G such that $H \leq K \leq H^{G,\infty}$, then K is contranormal in $H^{G,\infty}$. However, the subgroup H no longer has to be contranormal in K. Thus, we naturally arrive at the following important type of subgroups.

A subgroup H of a group G is said to be ***nearly abnormal in G***, if H is contranormal in every subgroup K including H [**KPSu2010**].

If H is an abnormal subgroup of a group G and K is a subgroup, containing H, then $x \in \langle H, H^x \rangle$ for every element $x \in K$. Taking into account the following obvious embedding $\langle H, H^x \rangle \leq H^{\langle x \rangle}$, we obtain that $H^K = K$. Therefore, every *abnormal subgroup is nearly abnormal.* In particular, every abnormal subgroup of a group G is contranormal in G. However not every contranormal subgroup is abnormal. The following simple example shows this.

Let P be a quasicyclic 2 – group. Consider the semidirect product $G = P \leftthreetimes \langle d \rangle$ where $|d| = 2$ and $d^{-1}ad = a^{-1}$ for any $a \in P$. Being hypercentral, G satisfies the normalizer condition. It follows that G has no proper abnormal subgroups. But $\langle d \rangle$ is a proper contranormal subgroup of G.

Let now H be a nearly abnormal subgroup of G, and consider an arbitrary subgroup K containing H. Suppose that $N_G(K) = L \neq K$. Then K is normal in L. It follows that $H = H^L \neq L$, so that H is not contranormal in L. This contradiction shows that $N_G(K) = K$. In other words, every nearly abnormal subgroup of G is weakly abnormal in G. Then from **Theorem P.5** we obtain the following result.

P.9. COROLLARY (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2010**]). *Let G be a hyper – Ñ – group and H be a subgroup of G. Then H is nearly abnormal in G if and only if H is abnormal.*

Conversely, let H be a weakly abnormal subgroup of G and let K be an arbitrary subgroup, containing H. Suppose that $H^K \neq K$. Subgroup H^K is normal in K, so that $N_K(H^K) = K \neq H^K$. Then $N_G(H^K) \neq H^K$, and H^K is self – normalizing. This contradiction shows that $H^K = K$. In other words, every weakly abnormal subgroup of G is nearly abnormal.

We already remarked that the Carter subgroups in finite soluble groups are important examples of abnormal subgroups. Some attempts of extending the definition of a Carter subgroup to infinite groups were made by S.E. Stonehewer [**StE1964**, **StE1965**], A.D. Gardiner, B. Hartley, M.J. Tomkinson [**GHT1971**], and M.R. Dixon [**DM1981**]. In those articles, some analogs of Carter subgroups in infinite groups have been considered. Note that all of those groups were locally finite. In [**KSu2005**], this concept have been extended to the class of not necessary periodic nilpotent – by – hypercentral groups.

We may define a Carter subgroup of a finite metanilpotent group as a minimal abnormal subgroup. The first logical step here is to consider the groups whose locally nilpotent residual is nilpotent.

Let G be a group, A be a normal subgroup of G. We say that A ***satisfies the condition Max – G*** (respectively ***Min – G***) if A satisfies the maximal (respectively the minimal) condition for G – invariant subgroups.

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an ***artinian – by – $\mathfrak{X}$ – group*** if G has a normal subgroup H such that $G/H \in \mathfrak{X}$ and H satisfies **Min** – G.

We will deal with artinian – by – hypercentral groups whose locally nilpotent residual is nilpotent. It is a natural first step. Since these groups are generalizations of finite metanilpotent groups, we will use for their definition some characterizations of Carter subgroups, which are valid for finite metanilpotent groups.

P.10. THEOREM (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2005**]). *Let G be an artinian – by – hypercentral group and suppose that its locally nilpotent residual K is nilpotent. Then G includes a minimal abnormal subgroup L. Moreover, L is a maximal hypercentral subgroup and it includes the upper hypercenter of G. In particular, G = KL. If H is another minimal abnormal subgroup, then H is conjugate to L.*

P.11. COROLLARY (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2005**]). *Let G be an artinian – by – hypercentral group and suppose that its locally nilpotent residual K is nilpotent. Then G includes a hypercentral abnormal subgroup L. Moreover, L is maximal hypercentral subgroup and it includes the upper hypercenter of G. In particular, G = KL. If H is another hypercentral abnormal subgroup, then H is conjugate to L.*

Let G be an artinian – by – hypercentral group with a nilpotent hypercentral residual. A subgroup H is called a ***Carter subgroup of a group G*** if H is a hypercentral abnormal subgroup of G (or, equivalently, H is a minimal abnormal subgroup of G).

A Carter subgroup in finite soluble group can be defined as a covering subgroup for the formation of nilpotent groups. As we will see, this characterization can be extended on the groups under consideration.

A subgroup H of a group G is said to be a ***$\mathbf{L}\mathfrak{N}$ – covering subgroup*** if H is locally nilpotent and if $S = HS^{\mathbf{L}\mathfrak{N}}$ for every subgroup S, which includes H. Recall that $S^{\mathbf{L}\mathfrak{N}}$ denotes the locally nilpotent residual of a subgroup S.

P.12. THEOREM (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2005**]). *Let G be an artinian – by – hypercentral group and suppose that its locally nilpotent residual K is nilpotent. If L is a Carter subgroup of G, then L is the $\mathbf{L}\mathfrak{N}$ – covering subgroup of G. Conversely, if H is a $\mathbf{L}\mathfrak{N}$ – covering subgroup of G, then H is a Carter subgroup of G.*

In a finite soluble group $\mathfrak{N}$ – covering subgroups are exactly $\mathfrak{N}$ – projectors. Therefore, a Carter subgroup of a finite soluble group can be defined as an $\mathfrak{N}$ – projector. This characterization also can be extended on the artinian – by – hypercentral groups.

A subgroup L of a group G is said to be a ***locally nilpotent projector of G***, if LH/H is a maximal locally nilpotent subgroup of G/H for each normal subgroup H of a group G.

P.13. THEOREM (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2005**]). *Let G be an artinian – by – hypercentral group and suppose that its locally nilpotent residual K is nilpotent. If L is a Carter subgroup of G, then L is a locally nilpotent projector of G. Conversely, if H is a locally nilpotent projector of G, then H is a Carter subgroup of G.*

For some narrower classes of groups, it is possible to get the traditional definition of the Carter subgroups.

Let G be a group and C be a normal subgroup of G. Then C is said to be a ***G – minimax*** if C has a finite series of G – invariant subgroups whose infinite factors are abelian and either satisfy **Min** – G or **Max** – G.

A group G is said to be ***generalized minimax***, if G is G – minimax. Every soluble minimax group is obviously generalized minimax. However, the class of generalized minimax groups is significantly broader than the class of soluble minimax groups.

P.14. THEOREM (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [**KOS2005**]). *Let G be a periodic generalized minimax group and suppose that its locally nilpotent residual K is nilpotent. If L is a self –normalizing locally nilpotent subgroup of G, then L is a* **$\mathfrak{LN}$ –** *covering subgroup of G. In other words, L is a Carter subgroup of G.*

Continuing with the situation where the upper normalizer series is very short, we come to another interesting sub – type.

Above, we introduced the concept of transitively normal subgroups. Note that one of the most important types of transitively normal subgroups was introduced by Ph. Hall as pronormal subgroups.

A subgroup H of a group G is said to be ***pronormal in G*** if for every $g \in G$ the subgroups H and H^g are conjugate in the subgroup $< H, H^g >$.

From the definition, it is clear that every abnormal subgroup is pronormal. On the other hand, every normal subgroup is also pronormal. Thus, pronormal subgroups have managed to combine these two antagonistic types of subgroups. Such important subgroups of finite soluble groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal.

As for normal subgroups, for the pronormal subgroup we consider the following weakened version, which has been introduced by M.S. Ba and Z.I. Borevich [BB1988].

Let G be a group. A subgroup H is called ***weakly pronormal in G*** if the subgroups H and H^x are conjugate in $H^{<x>}$ for each element x of a group G.

The inclusion $< H, H^x > \leq H^{<x>}$ shows that every pronormal subgroup is weak pronormal. Converse is not true (look at the corresponding example in [BB1988]).

Let G be a group and H be a subgroup of G. We say that H has the ***Frattini property*** if for every subgroups K, L such that $H \leq K$ and K is normal in L we have $L = N_L(H)K$.

P.15. THEOREM (M.S. Ba, Z.I. Borevich [**BB1988**]). *Let G be a group and H be a subgroup of G. Then:*

(i) H is weakly pronormal in G if and only if H has a Frattini property; in particular, every pronormal subgroup of G has a Frattini property;

(ii) if H is weakly pronormal in G, then H is weakly abnormal in G if and only if $H = N_G(H)$.

The following result establishes the relationships between weakly pronormal and pronormal subgroups.

P.16. THEOREM (M.S. Ba, Z.I. Borevich [**BB1988**]). *Let G be a group and H be a subgroup of G. Then H is pronormal in G if and only if H the following conditions hold:*

(i) H is weakly pronormal;

(ii) if L is a subgroup of G including H and g is an element of G such that $H \leq L^g$, *then there exists an element* $x \in N_G(H)$ *with the property* $L^x = L^g$.

P.17. PROPOSITION. *Let G be a group and H be a subgroup of G. Then:*

(i) if H is pronormal in G, then $N_G(H)$ *is abnormal in G;*

(ii) if H is pronormal in G, then H is abnormal if and only if $H = N_G(H)$.

For finite soluble groups, T.A. Peng obtained the following characterization of pronormal subgroups.

P.18. THEOREM (T.A. Peng [**PT1971**]). *Let G be a finite soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property. In other words, D is pronormal in G if and only if D is weakly pronormal in G.*

This Peng's characterization of pronormal subgroups could be extended to infinite groups in the following way.

P.19. THEOREM (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [**KOS2005**]). *Let G be a hyper – N – group and D be a subgroup of G. Then D is pronormal in G if and only if D has a Frattini property. In other words, D is pronormal in G if and only if D is weakly pronormal in G.*

As corollaries we obtain

P.20. COROLLARY (F. de Giovanni, G. Vincenzi [**deGV2001**]). *Let G be a hyperabelian group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

P.21. COROLLARY. *Let G be a soluble group and D be a subgroup of G. Then D is pronormal in G if and only if D has the Frattini property.*

If H is a pronormal subgroup of a group G and L is an intermediate subgroup for H, then H is pronormal in L, so **Proposition P.17** shows that $\mathbf{N}_L(H)$ is abnormal in L. We observe that every abnormal subgroup is contranormal.

A subgroup H of a group G is called ***nearly pronormal in G*** if $\mathbf{N}_L(H)$ is contranormal in L for every subgroup L, including H.

So we can see, that every pronormal subgroup is nearly pronormal, but converse is not true.

Let G be a special unitary group of 3 × 3 matrices over a field $\mathbf{F}_9$ of order 9. This group is simple and its order is 6048.

The multiplicative group $\mathbf{U}(\mathbf{F}_9)$ is cyclic and let g be an element such that $\langle g \rangle = \mathbf{U}(\mathbf{F}_9)$. Let K be a subgroup, generated by the following matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} g & g^2 & g^5 \\ g^5 & 0 & g^5 \\ g & g^6 & g^5 \end{pmatrix}, \begin{pmatrix} 1 & g^2 & 1 \\ g^2 & 0 & g^6 \\ 1 & g^6 & 1 \end{pmatrix}.$$

This group is nearly pronormal, but not pronormal in G. The order of K is 24; it is soluble, but not nilpotent. We observe that K is isomorphic to $\mathbf{S}_4$.

Nevertheless, for some classes of generalized soluble groups the concepts of nearly pronormal and pronormal subgroups coincide.

P.22. THEOREM (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2011**]). *Let G be a group having ascending series whose factors are abelian. Then every nearly pronormal subgroup of G is weakly pronormal in G.*

P.23. THEOREM (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2010**]). *Let G be a hyper – N – group. Then every nearly pronormal of G is pronormal.*

P.24. COROLLARY (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2010**]). *Let G be a soluble group. Then every nearly pronormal of G is pronormal.*

P.25. COROLLARY. *Let G be a soluble group. Suppose that a subgroup H satisfies the following condition: if K is a subgroup including H, then $N_K(H)$ is abnormal in K. Then K is pronormal in G.*

P.26. COROLLARY (G.J. Wood [**WG1974**]). *Let G be a finite soluble group. Suppose that a subgroup H satisfies the following condition: if K is a subgroup including H, then $N_K(H)$ is abnormal in K. Then K is pronormal in G.*

We observe also that for the generalized pronormal subgroups the class of an Ñ – group plays a special role. The following result justifies this statement.

P.27. THEOREM (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2010**]). *Let G be an Ñ – group and H be a subgroup of G. Then:*

(i) if H is weakly pronormal in G, then H is normal in G;

(ii) if H is nearly pronormal in G, then H is normal in G.

In particular, if H is pronormal in G, then H is normal in G.

P.28. COROLLARY (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2010**]). *Let G be a locally nilpotent group and H be a subgroup of G. Then:*

(i) if H is weakly pronormal in G, then H is normal in G;

(ii) if H is nearly pronormal in G, then H is normal in G.

P.29. COROLLARY (N.F. Kuzennyi, I.Ya. Subbotin [**KuS1988**]). *Let G be a locally nilpotent group and H be a subgroup of G. If H is pronormal in G, then H is normal in G.*

T.A. Peng has considered finite groups all of whose subgroups are pronormal. Its turned out that these groups are tightly connected to the groups, in which the relation «to be a normal subgroup» is transitive. More precisely, T.A. Peng proved the following result.

P.30. THEOREM (T.A. Peng [**PT1969**]). *Let G be a finite soluble group. Then every subgroup of G is pronormal if and only if G is a T – group.*

However, in the case of infinite groups, as shows the following theorem shows, the situation is more varied.

P.31. THEOREM (N.F. Kuzennyi, I.Ya. Subbotin [**KuS1988[1]**]). *Let G be a locally soluble group or a periodic locally graded group. Then the following conditions are equivalent:*

(i) every cyclic subgroup of G is pronormal in G;

(ii) G is a soluble $\overline{\boldsymbol{T}}$ *– group.*

Initially, the infinite groups whose subgroups are pronormal have been considered in [**KuS1987**]. The authors completely described such infinite locally soluble non – periodic and infinite locally graded periodic groups. The main result of their paper is the following theorem.

P.32. THEOREM (N.F. Kuzennyi, I.Ya. Subbotin [**KuS1987**]). *Let G be a group whose subgroups are pronormal, and L be a locally nilpotent residual of G. Then:*

(i) if G is periodic and locally graded, then G is a soluble $\overline{\boldsymbol{T}}$ *– group, in which L is a complement to every Sylow* $\boldsymbol{\Pi}(G/L)$ *– subgroup;*

(ii) if G is not periodic and locally soluble, then G is abelian.

Conversely, if G has a such structure, then every subgroup of G is pronormal in G.

N.F. Kuzennyi and I.Ya. Subbotin also completely described locally graded periodic groups in which all primary subgroups are pronormal [**KuS1989**], and infinite locally soluble groups in which all infinite subgroups are pronormal [**KuS1988[1]**]. They proved that in the infinite case, the class of groups whose subgroups are pronormal is a proper subclass of the class of groups with the transitivity of normality. Moreover, it is also a proper subclass of the class of groups whose primary subgroups are pronormal. However, the pronormality condition for all subgroups can be weakened to the pronormality for only abelian subgroups [**KuS1992**].

In the paper [**KRV2007**], the groups whose subgroups are nearly pronormal have been considered.

P.33. THEOREM (L.A. Kurdachenko, A. Russo, G. Vincenzi [**KRV2007**]). *Let G be a locally radical group. Then:*

(i) if every cyclic subgroup of G is nearly pronormal, then G is $\overline{\boldsymbol{T}}$ – group;

(ii) if every subgroup of G is nearly pronormal, then every subgroup of G is pronormal in G.

Theorems P.31, P.33 points out on the influence of cyclic pronormal and cyclic generalized pronormal subgroups on the structure of a group.

Note some other generalizations of pronormal subgroups. In the paper of S. Li and Z. Meng [**LM2007**], the authors introduced the following type of subgroups.

A subgroup H of a group G is called a ***self – conjugate permutable in G***, if H satisfies the following condition: an equality $HH^g = H^gH$ implies $H^g = H$.

We note at once that every pronormal subgroup D of a group G is self – conjugate permutable in G. Indeed, let g be an arbitrary element of G. Put $L = \langle D, D^g \rangle$. Since D is pronormal, there exists an element $x \in L$ such that $D^g = D^x$. Thus $L = \langle D, D^x \rangle$ where $x \in L$. Suppose now that D and D^g are permutable. Put $K = D^g = D^x$, then $L = DK$. Since $x \in L$, $x = dy$, where $d \in D$, $y \in K$. Then $K = D^x = D^{dy} = D^y$. It follows that $D^g = K = yKy^{-1} = y(y^{-1}Dy)y^{-1} = D$, which shows that D self – conjugate permutable in G.

However, converse is not true. The following example shows it. Again consider a special unitary group of 3 × 3 matrices over a field $\mathbb{F}_9$ of order 9. This group is simple and its order is 6048.

The multiplicative group $\mathbf{U}(\mathbb{F}_9)$ is cyclic and let g be an element such that $< g > = \mathbf{U}(\mathbb{F}_9)$. Let K be a subgroup, generated by the following matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & g^4 & g^4 \\ 1 & g^4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & g^4 \\ 0 & g^4 & 0 \\ g^4 & 0 & 0 \end{pmatrix}.$$

The subgroup K is self – conjugate permutable but not pronormal. Note that this subgroup is soluble but non – nilpotent.

It is not hard to see that every self – conjugate permutable subgroup is transitively normal. The following result is in contrast in some sense with **Theorem P.33.**

P.34. THEOREM (L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin [**KPSu2012**]). *Let G be a locally graded group. Then:*

(i) if G is not periodic, then every subgroup of G is self – conjugate permutable if and only if G is abelian;

(ii) if G is not periodic, then every subgroup of G is self – conjugate permutable if and only if G is a soluble $\overline{\boldsymbol{T}}$ – group.

P.35. COROLLARY (D.J.S. Robinson, A. Russo, G. Vincenzi [**RRV2007**]). *Let G be a locally graded group. If G is not periodic, then every subgroup of G is pronormal if and only if G is abelian.*

The following well – known characterizations of finite nilpotent groups are tightly bounded to abnormal and pronormal subgroups.

A finite group G is nilpotent if and only if G has no proper abnormal subgroups.

A finite group G is nilpotent if and only if its every pronormal subgroup is normal.

Note that since the normalizer of a pronormal subgroup is abnormal, *the absence of abnormal subgroups is equivalent to the normality of all pronormal subgroups.*

Theorem P.27 and **Corollary P.29** we can rewrite in the following way.

Let G be an Ñ – group. Then G has no proper abnormal subgroups.

Let G be a locally nilpotent group. Then G has no proper abnormal subgroups.

There exists an example of an Ñ – group, which is not locally nilpotent [**WJ1977**]. It follows that the absence of proper abnormal subgroups does not imply locally nilpotency. There is a natural question: in what groups the absence of proper abnormal subgroups is equivalent to their locally nilpotency? In other words, it would be interesting to get the criteria of nilpotency of a group in terms of pronormality and abnormality of its subgroups. In the paper [**KOS2002**], the first such nilpotency criterion was obtained.

P.36. THEOREM (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [**KOS2002**]). *Let G be a soluble generalized minimax group. If every pronormal subgroup of G is normal (or, what is equivalent, G has no proper abnormal subgroups), then G is hypercentral.*

The following criteria of hypercentrality have been obtained in [**KRV2006**].

P.37. THEOREM (L.A. Kurdachenko, A. Russo, G. Vincenzi [**KRV2006**]).

(i) Let G be a group whose pronormal subgroups are normal. Then every ***FC*** *– hypercenter of G having finite number is hypercentral.*

(ii) Let G be an ***FC*** *– nilpotent group. If all pronormal subgroups of G are normal, then G is hypercentral.*

(iii) Let G be a group whose pronormal subgroups are normal. Suppose that H is an ***FC*** *– hypercenter of G having finite number. If C is a normal subgroup of G such that $C \geq H$ and C/H is hypercentral, then C is a hypercentral.*

For periodic groups, the above results were obtained in [**KSu2003**].

Observe that abnormal subgroups are an important particular case of contranormal subgroups: abnormal subgroups are exactly the subgroups that are contranormal in each subgroup containing them. On the other hand, abnormal subgroups are a particular case of pronormal subgroups.

Pronormal subgroups are connected to contranormal subgroups in the following way. If H is a pronormal subgroup of a group G and $H \leq K$, then $\mathbf{N}_K(H)$ is an abnormal, and hence contranormal, subgroup of K.

The next results connects the conditions of generalized nilpotency to descendant subgroups.

P.38. THEOREM (L.A. Kurdachenko, I.Ya. Subbotin [**KSu2003**]). *Let G be a group, every subgroup of which is descendant. If G is* ***FC*** *–hypercentral, then G is hypercentral.*

P.39. THEOREM (L.A. Kurdachenko, J. Otal, I.Ya. Subbotin [**KOS2005**]). *Let G be a soluble generalized minimax group. Then every subgroup of G is descendant if and only if G is nilpotent.*

If every subgroup of a group G is descendant, then G does not include proper contranormal subgroups. The study of groups without proper contranormal subgroups is the next logical step. We observe that every non – normal maximal subgroup of an arbitrary group is contranormal. Since a finite group whose maximal subgroups are normal is nilpotent, we come to the following criterion of nilpotency of finite groups in terms of contranormal subgroups.

A finite group G is nilpotent if and only if G does not include proper contranormal subgroups.

The question about an analog of this criterion for infinite groups is very natural. However, in general, the absence of proper contranormal subgroups does not imply nilpotency. In fact, there exist non – nilpotent groups, all subgroups of which are subnormal. The first such example has been constructed by H. Heineken and I.J. Mohamed [**HM1972**]. Nevertheless, for some classes of infinite groups the absence of proper contranormal subgroups does imply nilpotency of a group. The groups without proper contranormal subgroups have been considered in the papers [**KOS2009[1]**, **KOS2009[2]**]. We show the main results of these articles.

P.40. THEOREM. *Let G be a group and H be a normal soluble – by – finite subgroup such that G/H is nilpotent. Suppose that H satisfies* ***Min*** *– G. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if soluble – by – finite group G without proper contranormal subgroups satisfies minimal condition for normal subgroups, then G is nilpotent.*

P.41. THEOREM. *Let G be a group and H be a normal Chernikov subgroup such that G/H is nilpotent. If G has no proper contranormal subgroups, then G is nilpotent. In particular, Chernikov group without proper contranormal subgroups is nilpotent.*

P.42. THEOREM. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian minimax subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is hyperabelian minimax group without proper contranormal subgroups, then G is nilpotent.*

P.43. THEOREM. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a Chernikov subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is a Chernikov group without proper contranormal subgroups, then G is nilpotent.*

P.44. THEOREM. *Let G be a group and let C be a normal soluble subgroup of G such that G/C is nilpotent. Suppose that C is a hyperabelian finitely generated subgroup. If G has no proper contranormal subgroups, then G is nilpotent. In particular, if G is hyperabelian finitely generated group without proper contranormal subgroups, then G is nilpotent.*

P.45. THEOREM. *Suppose that the group G includes a normal G–minimax subgroup C such that G/C is a nilpotent group of finite section rank. If G has no proper contranormal subgroups, then G is nilpotent.*

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