Large orbits of soluble linear groups and Gluck's conjecture

Hangyang Meng

Shanghai University
E-mail: hymeng2009@shu.edu.cn

Advances in group theory and applications
07-06-2021

PhD thesis:

Regular orbits of actions of finite soluble groups. Applications,

- Meng, H.; Ballester-Bolinches, A.; Esteban-Romero, R. On large orbits of subgroups of linear groups. Trans. Amer. Math. Soc. 372 (2019), no. 4, 2589-2612.
- Meng, H.; Ballester-Bolinches, A.; Esteban-Romero, R. On large orbits of supersoluble subgroups of linear groups. J. Lond. Math. Soc. (2) 101 (2020), no. 2, 490-504.

All sets, groups, fields and modules to be considered are finite.

Let G be a group.

- $\mathrm{Irr}(G)$ the set of all irreducible complex character of G;
- $b(G) = \max\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$.

D. Gluck showed that if G is soluble, then $|G : \mathrm{F}(G)| \leq b(G)^{7.5}$ and conjectured:

Gluck's Conjecture, 1985

$|G : \mathrm{F}(G)| \leq b(G)^2$ holds for all soluble groups G.

- D. Gluck. The largest irreducible character degree of a finite group. Canad. J. Math., 37(3):442–415, 1985.

A linear bound would fail!!!

Example

Let $G_n = A\Gamma(p^n)$ be the affine semilinear group on the field of p^n elements, p a prime number. Then $b(G) = p^n - 1$ and

$$[G_n : \mathrm{F}(G_n)] = n(p^n - 1) = nb(G_n).$$

For every choice of constants A and B, there is a G_n such that $[G_n : \mathrm{F}(G_n)] > Ab(G_n) + B$.

$A\Gamma(p^n)$: the set of all maps $f : \mathrm{GF}(p^n) \to \mathrm{GF}(p^n)$ such that

$$f(x) = bx^{\sigma} + a, x \in \mathrm{GF}(p^n).$$

for some $b \in \mathrm{GF}(p^n)^{\times}, a \in \mathrm{GF}(p^n), \sigma \in Aut(\mathrm{GF}(p^n))$.

The Structure of $A\Gamma(p^n)$

$G := A\Gamma(p^n)$;
A : the set of all maps $x \to x + a, a \in \mathrm{GF}(p^n)$;
B : the set of all maps $x \to bx, b \in \mathrm{GF}(p^n)^\times$;
C : the set of all maps $x \to x^\sigma, \sigma \in Aut(\mathrm{GF}(p^n))$.

- $G = A \rtimes (B \rtimes C)$;
- B acts regularly on $A - \{1\}$;
- $|G : \mathrm{C}_G(a)| = |B|, 1 \neq a \in A$;
- $\Gamma(p^n) := BC$, the semilinear group on $\mathrm{GF}(p^n)$.

Dual Groups

- If A is an abelian group, $\mathrm{Irr}(A) = \mathrm{Hom}(A, \mathbb{C}^\times)$ is a group under the product

$$(\chi\psi)(a) = \chi(a)\psi(a), \chi, \psi \in Irr(A), a \in A.$$

which is called the dual group of A, also denote by $A^\star$.

- Suppose that a group X acts on an abelian group A. Then we can define an action of X on the dual group $A^\star$ by setting

$$\chi^h(a) = \chi(a^{h^{-1}}), \chi \in \mathrm{Irr}(A), h \in X, a \in A.$$

Reduction

- By induction, $\Phi(G) = 1$.
- Set $V = \mathrm{F}(G)$ and there exists $X \leq G$ such that $G = VX$ and $V \cap X = 1$; Moreover, V is a faithful, completely reducible X-module.
- Let $U = \mathrm{Irr}(V)$. Then U is also a faithful, completely reducible X-module;
- **If there exists** $\lambda \in U$ **such that** $|X : \mathrm{C}_X(\lambda)| \geq |X|^{\frac{1}{2}}$, then we can use Clifford Correspondence to prove Gluck's conjecture directly.

Hypothesis

- G is a soluble group;
- V is a faithful completely reducible G-module, possibly of mixed characteristic.

Gluck's Strategy

- Does G have an orbit on V of length at least $|G|^{\frac{1}{2}}$??

Definition

Let a group G act on a set X. A G-orbit $\mathcal{O}$ on X is called *regular* if $\mathrm{C}_G(w) \triangleq \{g \in G : w^g = w\} = 1, w \in \mathcal{O}$.

- G has n regular orbits on X, i.e., there exist n different regular G-orbits on X;
- If G has one regular orbit on X, then G acts faithfully on X.

Interesting Problem

Does G always have one regular orbit on $V \oplus V$?

- If G has one regular orbit on $V \oplus V$, i.e., there exists $(u, v) \in V \oplus V$ such that $\mathrm{C}_G(u) \cap \mathrm{C}_G(v) = 1$, then

$$\min\{|\mathrm{C}_G(u)|, |\mathrm{C}_G(v)|\} \leq |G|^{\frac{1}{2}},$$

  that is, there exists a G-orbit on V whose length is at least $|G|^{\frac{1}{2}}$.
- The answer is negative.

Some Known Results about Regular Orbits

Suppose that (G, V) satisfies **Hypothesis**.

- If $|[V]G|$ is odd, then G has one regular orbit on $V \oplus V$. (A. Espuelas, 1991, Illinois J. Math.)
- If the Sylow 2-subgroup of $[V]G$ is abelian, then G has two regular orbits on $V \oplus V$. (S. Dolfi and E. Jabara, 2007, J. Algebra)
- If 3 does not divide $|G|$, then G has three regular orbits on $V \oplus V$. (Y. Yang, 2011, Proc. Amer. Math. Soc.)
- If G is nilpotent, then G has three regular orbits on $V \oplus V$. (A. Moretó and T. R. Wolf, 2004, Adv. Math.)
- If G is supersoluble, then G has one regular orbit on $V \oplus V$. (T. R. Wolf, 1999, J. Algebra)
- If $(|G|, |V|) = 1$, then G has one regular orbit on $V \oplus V$. (S. Dolfi, 2008, Trans. Amer. Math. Soc.)

Primitive modules

An irreducible G-module V is called *imprimitive* if V can be written $V = V_1 \oplus ... \oplus V_n$ for $n > 1$ subspaces V_i that are permuted (naturally transitively) by G. Moreover, V is called *primitive* if V is not imprimitive.

Primitive Case: Dolfi ,2008

Let G be a soluble primitive subgroup of $GL(n, p)$, p a prime number, n a positive integer, and let V be the natural module for G. Then G has at least p regular orbits on $V \oplus V$, unless G is one of the following groups:

- $GL(2, 2)$;
- $SL(2, 3)$ or $GL(2, 3)$;
- $3^{1+2}.SL(2, 3)$ or $3^{1+2}.GL(2, 3) \leq GL(6, 2)$;
- $(Q_8 * Q_8).K \leq GL(4, 3)$ where K is isomorphic to a subgroup of index 1, 2 or 4 of $O^+(4, 2)$.

In a recent paper, Y. Yang extends some of these results to the subgroup case.

- Y. Yang. Large orbits of subgroups of solvable linear groups. Israel J. Math., 199(1):345–362, 2014.

Theorem (Yang, 2014, Israel J. Math.)

Suppose that (G, V) satisfies **Hypothesis** and $H \leq G$.

- If H is nilpotent, then H has three regular orbits on $V \oplus V$.
- If 3 does not divide $|H|$, then H has three regular orbits on $V \oplus V$.
- If the Sylow 2-subgroup of $[V]H$ is abelian, then H has two regular orbits on $V \oplus V$.

- Let X be a group. A group G is said to be X-*free* if X cannot be obtained as a quotient of a subgroup of G;

Analysis

- H is nilpotent or $3 \nmid |H| \Rightarrow H$ is S_3-free $\Rightarrow [V]H$ is S_4-free;
- The Sylow 2-subgroup of $[V]H$ is abelian $\Rightarrow [V]H$ is S_4-free.

Theorem (Meng, Ballester, Esteban, 2019)

Suppose that (G, V) satisfies **Hypothesis** and let H be a subgroup of G such that $[V]H$ is S_4-free. Then H has two regular orbits on $V \oplus V$. Furthermore, if H is $\Gamma(2^3)$-free and $\mathrm{SL}(2,3)$-free, then H has three regular orbits on $V \oplus V$.

- Cover all known results except Wolf's supersoluble case;
- Meng, H. ; Ballester-Bolinches, A. ; Esteban-Romero, R. On large orbits of subgroups of linear groups.Trans. Amer. Math. Soc. 372 (2019), no. 4, 2589–2612.

Thus we also extends Wolf's result to the subgroup case as follows:

Theorem (Meng, Ballester, Esteban, 2020)

Suppose that (G, V) satisfies **Hypothesis** and let H be a supersoluble subgroup of G. Then H has one regular orbit on $V \oplus V$.

- Meng, H.; Ballester-Bolinches, A.; Esteban-Romero, R. On large orbits of supersoluble subgroups of linear groups. J. Lond. Math. Soc. (2) 101 (2020), no. 2, 490–504.

Example

Let $K = \mathrm{GL}(2,2)$ and $W = \mathrm{GF}(2) \oplus \mathrm{GF}(2)$ the natural faithful module of K over $\mathrm{GF}(2)$. Let $S \cong S_2$ be a permutation group on $\Omega = \{1, 2\}$. Write $G = K \wr S$ and $V = W^{\Omega}$, the set of all maps from Ω to W. Then V is a faithful, irreducible G-module. Set

$$H = \{(f, \sigma) \in G : f \in K, \sigma \in S, f(1) = f(2)\} \cong S_3 \times C_2.$$

Then we have:

- G is soluble and H is supersoluble;
- V is not a completely reducible H-module;
- G does not have any regular orbits on $V \oplus V$ but H does;
- $[V]H$ has a subgroup isomorphic to S_4.

- The action of H on V is not completely reducible in general.

Theorem (Meng, Ballester, Esteban, 2019)

Let G be a soluble group satisfying one of the following conditions:

- G is S_4-free;
- $G/\mathrm{F}(G)$ is S_4-free and $\mathrm{F}(G)$ is of odd order;
- $G/\mathrm{F}(G)$ is S_3-free;

Then Gluck's conjecture is true for G.

- Meng, H. ; Ballester-Bolinches, A. ; Esteban-Romero, R. On large orbits of subgroups of linear groups. Trans. Amer. Math. Soc. 372 (2019), no. 4, 2589–2612.

Corollary (S. Dolfi and E. Jabara; J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen)

Let G be a soluble group. If either the Sylow 2-subgroups of G are abelian or $|G/\mathrm{F}(G)|$ is not divisible by 6, then Gluck's conjecture is true for G.

- S. Dolfi and E. Jabara. Large character degrees of solvable groups with abelian Sylow 2-subgroups. J. Algebra, 313(2):687–694, 2007.
- J. P. Cossey, Z. Halasi, A. Maróti, and H. N. Nguyen. On a conjecture of Gluck. Math. Z., 279:1067–1080, 2015.

Gluck's Strategy does not work in some cases!!!

Example

Let $K = \mathrm{GL}(2,2)$ and $W = \mathrm{GF}(2) \oplus \mathrm{GF}(2)$ the natural faithful module of K over GF(2). Let $S \cong S_3$ be a permutation group on $\Omega = \{1,2,3\}$. Write $G = K \wr S$ and $V = W^{\Omega}$, the set of all maps from Ω to W. Then V is a faithful, irreducible G-module. Then we have:

- G has no regular orbit on $V \oplus V$;
- G has no orbit as large as $|G|^{\frac{1}{2}}$;(Orbit size:1,9,27,27)
- But G satisfies Gluck's conjecture.

p-local Gluck's conjecture

Define

$$b_p(G) = max\{\chi(1)_p : \chi \in \mathrm{Irr}(G)\}$$

to be the greatest p-part of the degrees of irreducible characters of G.

p-local Gluck's conjecture

Theorem(Espuelas, Navarro, 1992, PAMS)

Let G be an odd order group and p is prime.

$$|G : \mathrm{F}(G)|_p \leq b_p(G)^2.$$

This bound can be reached!

Let G be the subgroup of index 2 in $A\Gamma(7^3)$.

$$|G : \mathrm{F}(G)|_3 = b_3(G)^2 = 3^2$$

- Espuelas, Alberto; Navarro, Gabriel. Blocks of small defect. Proc. Amer. Math. Soc. 114 (1992), no. 4, 881–885.

Theorem(Yang, 2015, JA)

Let G be a soluble group and $p \geq 5$ is prime.

$$|G : \mathrm{F}(G)|_p \leq b_p(G)^{\frac{5}{2}}.$$

- Yang, Yong . Blocks of small defect. J. Algebra 429 (2015), 192–212.

Thanks for your attention!.